Contents 6

Exponential and Logarithmic Functions

Learning outcomes

In this Workbook you will learn about one of the most important functions in mathematics, science and engineering - the exponential function. You will learn how to combine exponential functions to produce other important functions, the hyperbolic functions, which are related to the trigonometric functions.

You will also learn about logarithms and the logarithmic function which is the function inverse to the exponential function. Finally you will learn what a log-linear graph is and how it can be used to simplify the presentation of certain kinds of data.

The Exponential Function

6.1

Introduction

In this Section we revisit the use of exponents. We consider how the expression a^x is defined when a is a positive number and x is *irrational*. Previously we have only considered examples in which x is a *rational* number. We consider these exponential functions $f(x) = a^x$ in more depth and in particular consider the special case when the base a is the exponential constant e where :

$$\text{e} = 2.7182818\ldots$$

We then examine the behaviour of e^x as $x \to \infty$, called **exponential growth** and of e^{-x} as $x \to \infty$ called **exponential decay**.

Prerequisites

Before starting this Section you should . . .

- have a good knowledge of indices and their laws
- have knowledge of rational and irrational numbers

Learning Outcomes

On completion you should be able to . . .

- approximate a^x when x is irrational
- describe the behaviour of a^x: in particular the exponential function e^x
- understand the terms **exponential growth** and **exponential decay**

1. Exponents revisited

We have seen (in HELM 1.2) the meaning to be assigned to the expression a^p where a is a positive number. We remind the reader that 'a' is called the **base** and 'p' is called the **exponent** (or power or index). There are various cases to consider:

If m, n are positive integers

- $a^n = a \times a \times \cdots \times a$ with n terms
- $a^{1/n}$ means the n^{th} root of a. That is, $a^{1/n}$ is that positive number which satisfies

  $(a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n}) = a$

  where there are n terms on the left hand side.
- $a^{m/n} = (a^{1/n}) \times (a^{1/n}) \times \cdots \times (a^{1/n})$ where there are m terms.
- $a^{-n} = \dfrac{1}{a^n}$

For convenience we again list the basic laws of exponents:

Key Point 1

$$a^m a^n = a^{m+n} \qquad \frac{a^m}{a^n} = a^{m-n} \qquad (a^m)^n = a^{mn}$$

$$a^1 = a, \qquad \text{and if} \quad a \neq 0 \qquad a^0 = 1$$

Example 1

Simplify the expression $\dfrac{p^{n-2}p^m}{p^3 p^{2m}}$

Solution

First we simplify the numerator:

$$p^{n-2}p^m = p^{n-2+m}$$

Next we simplify the denominator:

$$p^3 p^{2m} = p^{3+2m}$$

Now we combine them and simplify:

$$\frac{p^{n-2}p^m}{p^3 p^{2m}} = \frac{p^{n-2+m}}{p^{3+2m}} = p^{n-2+m}p^{-3-2m} = p^{n-2+m-3-2m} = p^{n-m-5}$$

Task Simplify the expression $\dfrac{b^{m-n}b^3}{b^{2m}}$

First simplify the numerator:

Your solution

$b^{m-n}b^3 =$

Answer

$b^{m-n}b^3 = b^{m+3-n}$

Now include the denominator:

Your solution

$$\frac{b^{m-n}b^3}{b^{2m}} = \frac{b^{m+3-n}}{b^{2m}} =$$

Answer

$$\frac{b^{m+3-n}}{b^{2m}} = b^{m+3-n-2m} = b^{3-m-n}$$

Task Simplify the expression $\dfrac{(5a^m)^2a^2}{(a^3)^2}$

Simplify the numerator:

Your solution

$(5a^m)^2a^2 =$

Answer

$(5a^m)^2a^2 = 25a^{2m}a^2 = 25a^{2m+2}$

Now include the denominator:

Your solution

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} =$$

Answer

$$\frac{(5a^m)^2a^2}{(a^3)^2} = \frac{25a^{2m+2}}{a^6} = 25a^{2m+2-6} = 25a^{2m-4}$$

a^x when x is any real number

So far we have given the meaning of a^p where p is an integer or a rational number, that is, one which can be written as a quotient of integers. So, if p is rational, then

$$p = \frac{m}{n} \qquad \text{where } m, n \text{ are integers}$$

Now consider x as a real number which cannot be written as a rational number. Two common examples of these **irrational** numbers are $\sqrt{2}$ and π. What we shall do is *approximate* x by a rational number by working to a fixed number of decimal places. For example if

$$x = 3.14159265\ldots$$

then, if we are working to 3 d.p. we would write

$$x \approx 3.142$$

and **this** number can certainly be expressed as a rational number:

$$x \approx 3.142 = \frac{3142}{1000}$$

so, in this case

$$a^x = a^{3.14159\ldots} \approx a^{3.142} = a^{\frac{3142}{1000}}$$

and the final term: $a^{\frac{3142}{1000}}$ can be determined in the usual way by calculator. Henceforth we shall therefore assume that the expression a^x is defined for all positive values of a and for **all** real values of x.

Task

By working to 3 d.p. find, using your calculator, the value of $3^{\pi/2}$.

First, approximate the value of $\dfrac{\pi}{2}$:

Your solution

$\dfrac{\pi}{2} \approx$ to 3 d.p.

Answer

$$\frac{\pi}{2} \approx \frac{3.1415927\ldots}{2} = 1.5707963\cdots \approx 1.571$$

Now determine $3^{\pi/2}$:

Your solution

$3^{\pi/2} \approx$

Answer

$3^{\pi/2} \approx 3^{1.571} = 5.618$ to 3 d.p.

2. Exponential functions

For a fixed value of the base a the expression a^x clearly varies with the value of x: it is a function of x. We show in Figure 1 the graphs of $(0.5)^x, (0.3)^x, 1^x, 2^x$ and 3^x.

The functions a^x (as different values are chosen for a) are called **exponential functions**. From the graphs we see (and these are true for *all* exponential functions):

If $a > b > 0$ then

$a^x > b^x$ if $x > 0$ and $a^x < b^x$ if $x < 0$

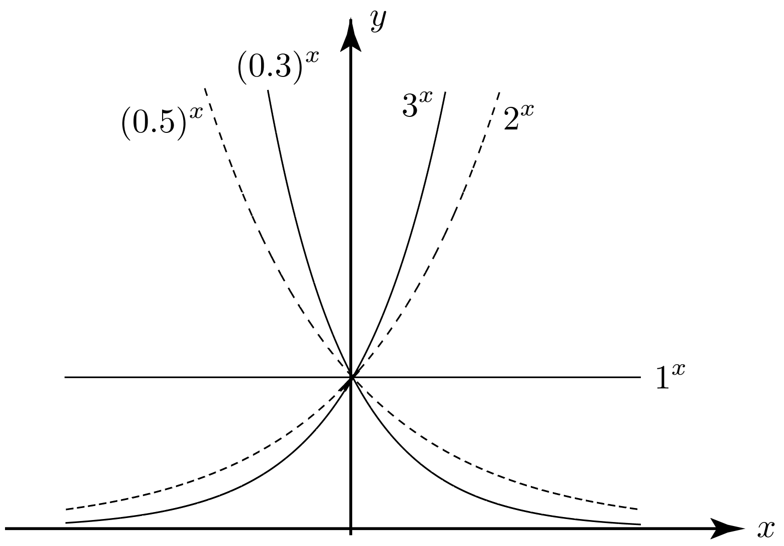


Figure 1: $y = a^x$ for various values of a

The most important and widely used exponential function has the particular base e $= 2.7182818\ldots$. It will not be clear to the reader why this particular value is so important. However, its importance will become clear as your knowledge of mathematics increases. The number e is as important as the number π and, like π, is also irrational. The approximate value of e is stored in most calculators. There are numerous ways of calculating the value of e. For example, it can be shown that the value of e is the end-point of the sequence of numbers:

$$\left(\frac{2}{1}\right)^1, \quad \left(\frac{3}{2}\right)^2, \quad \left(\frac{4}{3}\right)^3, \quad \ldots, \quad \left(\frac{17}{16}\right)^{16}, \quad \ldots, \quad \left(\frac{65}{64}\right)^{64}, \ldots$$

which, in decimal form (each to 6 d.p.) are

$$2.000000, \quad 2.250000, \quad 2.370370, \quad \ldots, \quad 2.637929, \quad \ldots, 2.697345, \quad \ldots$$

This is a slowly converging sequence. However, it does lead to a precise definition for the value of e:

$$\mathrm{e} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n$$

An quicker way of calculating e is to use the (infinite) series:

$$\mathrm{e} = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots + \frac{1}{n!} + \ldots$$

where, we remember,

$$n! = n \times (n-1) \times (n-2) \times \ldots (3) \times (2) \times (1)$$

(This is discussed more fully in HELM 16: Sequences and Series.)

Although all functions of the form a^x are called exponential functions we usually refer to e^x as *the* exponential function.

Key Point 2

e^x is **the** exponential function where $\mathrm{e} = 2.71828\ldots$

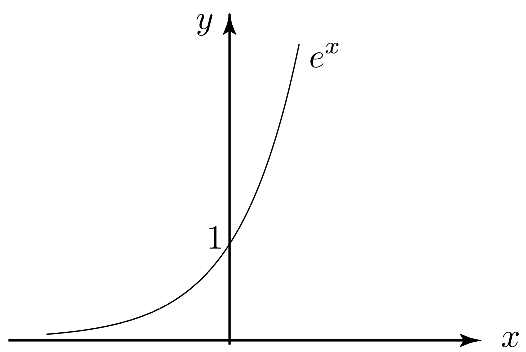


Figure 2: $y = \mathrm{e}^x$

Exponential functions (and variants) appear in various areas of mathematics and engineering. For example, the shape of a hanging chain or rope, under the effect of gravity, is well described by a combination of the exponential curves e^{kx}, e^{-kx}. The function e^{-x^2} plays a major role in statistics; it being fundamental in the important normal distribution which describes the variability in many naturally occurring phenomena. The exponential function e^{-kx} appears directly, again in the area of statistics, in the Poisson distribution which (amongst other things) is used to predict the number of events (which occur randomly) in a given time interval.

From now on, when we refer to an exponential function, it will be to the function e^x (Figure 2) that we mean, unless stated otherwise.

Task

Use a calculator to determine the following values correct to 2 d.p.
(a) $e^{1.5}$, (b) e^{-2}, (c) e^{17}.

Your solution

(a) $e^{1.5} =$ (b) $e^{-2} =$ (c) $e^{17} =$

Answer

(a) $e^{1.5} = 4.48$, (b) $e^{-2} = 0.14$, (c) $e^{17} = 2.4 \times 10^7$

Task

Simplify the expression $\dfrac{e^{2.7}e^{-3(1.2)}}{e^2}$ and determine its numerical value to 3 d.p.

First simplify the expression:

Your solution

$$\frac{e^{2.7}e^{-3(1.2)}}{e^2} =$$

Answer

$$\frac{e^{2.7}e^{-3(1.2)}}{e^2} = e^{2.7}e^{-3.6}e^{-2} = e^{2.7-3.6-2} = e^{-2.9}$$

Now evaluate its value to 3 d.p.:

Your solution

$e^{-2.9} =$

Answer

0.055

3. Exponential growth

If $a > 1$ then it can be shown that, no matter how large K is:

$$\frac{a^x}{x^K} \to \infty \quad \text{as} \quad x \to \infty$$

That is, if K is fixed (though chosen as large as desired) then eventually, as x increases, a^x will become larger than the value x^K provided $a > 1$. The growth of a^x as x increases is called **exponential growth**.

Task

A function $f(x)$ grows exponentially and is such that $f(0) = 1$ and $f(2) = 4$. Find the exponential curve that fits through these points. Assume the function is $f(x) = \mathrm{e}^{kx}$ where k is to be determined from the given information. Find the value of k.

First, find $f(0)$ and $f(2)$ by substituting in $f(x) = \mathrm{e}^{kx}$:

Your solution

When $x = 0 \quad f(0) = \mathrm{e}^0 = 1$

When $x = 2, \quad f(2) = 4 \quad$ so $\mathrm{e}^{2k} = 4$

By trying values of k: $0.6, 0.7, 0.8$, find the value such that $\mathrm{e}^{2k} \approx 4$:

Your solution

$\mathrm{e}^{2(0.6)} =$ $\qquad$ $\mathrm{e}^{2(0.7)} =$ $\qquad$ $\mathrm{e}^{2(0.8)} =$

Answer

$\mathrm{e}^{2(0.6)} = 3.32$ (too low) $\qquad$ $\mathrm{e}^{2(0.7)} = 4.055$ (too high)

Now try values of k: $k = 0.67,\ 0.68,\ 0.69$:

Your solution

$\mathrm{e}^{2(0.67)} =$ $\qquad$ $\mathrm{e}^{2(0.68)} =$ $\qquad$ $\mathrm{e}^{2(0.69)} =$

Answer

$\mathrm{e}^{2(0.67)} = 3.819$ (low) $\qquad$ $\mathrm{e}^{2(0.68)} = 3.896$ (low) $\qquad$ $\mathrm{e}^{2(0.69)} = 3.975$ (low)

Next try values of $k = 0.691,\ 0.692$:

Your solution

$\mathrm{e}^{2(0.691)} =$ $\qquad$ $\mathrm{e}^{2(0.692)} =$ $\qquad$ $\mathrm{e}^{2(0.693)} =$

Answer

$\mathrm{e}^{2(0.691)} = 3.983$, (low) $\qquad$ $\mathrm{e}^{2(0.692)} = 3.991$ (low) $\qquad$ $\mathrm{e}^{2(0.693)} = 3.999$ (low)

Finally, state the exponential function with k to 3 d.p. which most closely satisfies the conditions:

Your solution

$y =$

Answer

The exponential function is $\mathrm{e}^{0.693x}$.

We shall meet, in Section 6.4, a much more efficient way of finding the value of k.

4. Exponential decay

As we have noted, the behaviour of e^x as $x \to \infty$ is called exponential growth. In a similar manner we characterise the behaviour of the function e^{-x} as $x \to \infty$ as **exponential decay**. The graphs of e^x and e^{-x} are shown in Figure 3.

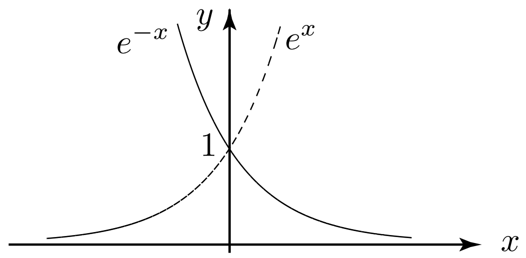


Figure 3: $y = e^x$ and $y = e^{-x}$

Exponential growth is very rapid and similarly exponential decay is also very rapid. In fact e^{-x} tends to zero so quickly as $x \to \infty$ that, no matter how large the constant K is,

$$x^K e^{-x} \to 0 \quad \text{as} \quad x \to \infty$$

The next Task investigates this.

Task

Choose $K = 10$ in the expression $x^K e^{-x}$ and calculate $x^K e^{-x}$ using your calculator for $x = 5,\ 10,\ 15,\ 20,\ 25,\ 30,\ 35$.

Your solution

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$							

Answer

x	5	10	15	20	25	30	35
$x^{10}e^{-x}$	6.6×10^4	4.5×10^5	1.7×10^5	2.1×10^4	1324	55	1.7

The topics of exponential growth and decay are explored further in Section 6.5.

Exercises

1. Find, to 3 d.p., the values of

   (a) 2^{-8} (b) $(5.1)^4$ (c) $(2/10)^{-3}$ (d) $(0.111)^6$ (e) $2^{1/2}$ (f) π^π (g) $e^{\pi/4}$ (h) $(1.71)^{-1.71}$

2. Plot $y = x^3$ and $y = e^x$ for $0 < x < 7$. For which integer values of x is $e^x > x^3$?

Answers

1. (a) 0.004 (b) 676.520 (c) 125 (d) 0.0 (e) 1.414 (f) 36.462 (g) 2.193 (h) 0.400
2. For integer values of x, $e^x > x^3$ if $x \geq 5$

The Hyperbolic Functions

6.2

Introduction

The hyperbolic functions $\sinh x,\ \cosh x,\ \tanh x$ etc are certain combinations of the exponential functions e^x and e^{-x}. The notation implies a close relationship between these functions and the trigonometric functions $\sin x,\ \cos x,\ \tan x$ etc. The close relationship is algebraic rather than geometrical. For example, the functions $\cosh x$ and $\sinh x$ satisfy the relation

$$\cosh^2 x - \sinh^2 x \equiv 1$$

which is very similar to the trigonometric identity $\cos^2 x + \sin^2 x \equiv 1$. (In fact every trigonometric identity has an equivalent hyperbolic function identity.)

The hyperbolic functions are not introduced because they are a mathematical nicety. They arise naturally and sufficiently often to warrant sustained study. For example, the shape of a chain hanging under gravity is well described by $\cosh$ and the deformation of uniform beams can be expressed in terms of $\tanh$.

Prerequisites

Before starting this Section you should ...

- have a good knowledge of the exponential function
- have knowledge of odd and even functions
- have familiarity with the definitions of $\tan,\ \sec,\ \mathrm{cosec}, \cot$ and of trigonometric identities

Learning Outcomes

On completion you should be able to ...

- explain how hyperbolic functions are defined in terms of exponential functions
- obtain and use hyperbolic function identities
- manipulate expressions involving hyperbolic functions

1. Even and odd functions

Constructing even and odd functions

A given function $f(x)$ can always be split into two parts, one of which is even and one of which is odd. To do this write $f(x)$ as $\frac{1}{2}[f(x) + f(x)]$ and then simply add and subtract $\frac{1}{2}f(-x)$ to this to give

$$f(x) = \frac{1}{2}[f(x) + f(-x)] + \frac{1}{2}[f(x) - f(-x)]$$

The term $\frac{1}{2}[f(x) + f(-x)]$ is **even** because when x is replaced by $-x$ we have $\frac{1}{2}[f(-x) + f(x)]$ which is the same as the original. However, the term $\frac{1}{2}[f(x) - f(-x)]$ is **odd** since, on replacing x by $-x$ we have $\frac{1}{2}[f(-x) - f(x)] = -\frac{1}{2}[f(x) - f(-x)]$ which is the negative of the original.

Example 2

Separate $x^3 + 2^x$ into odd and even parts.

Solution

$$f(x) = x^3 + 2^x$$

$$f(-x) = (-x)^3 + 2^{-x} = -x^3 + 2^{-x}$$

Even part:

$$\frac{1}{2}(f(x) + f(-x)) = \frac{1}{2}(x^3 + 2^x - x^3 + 2^{-x}) = \frac{1}{2}(2^x + 2^{-x})$$

Odd part:

$$\frac{1}{2}(f(x) - f(-x)) = \frac{1}{2}(x^3 + 2^x + x^3 - 2^{-x}) = \frac{1}{2}(2x^3 + 2^x - 2^{-x})$$

Task

Separate the function $x^2 - 3^x$ into odd and even parts.

First, define $f(x)$ and find $f(-x)$:

Your solution

$f(x) =$ $\qquad\qquad$ $f(-x) =$

Answer

$f(x) = x^2 - 3^x, \quad f(-x) = x^2 - 3^{-x}$

Now construct $\frac{1}{2}[f(x)+f(-x)]$, $\frac{1}{2}[f(x)-f(-x)]$:

Your solution

$\frac{1}{2}[f(x)+f(-x)] =$ $\qquad$ $\frac{1}{2}[f(x)-f(-x)] =$

Answer

$$\frac{1}{2}[f(x)+f(-x)] = \frac{1}{2}(x^2-3^x+x^2-3^{-x})$$
$$= x^2-\frac{1}{2}(3^x+3^{-x}).$$ This is the even part of $f(x)$.

$$\frac{1}{2}[f(x)-f(-x)] = \frac{1}{2}(x^2-3^x-x^2+3^{-x})$$
$$= \frac{1}{2}(3^{-x}-3^x).$$ This is the odd part of $f(x)$.

The odd and even parts of the exponential function

Using the approach outlined above we see that the even part of e^x is

$$\frac{1}{2}(\mathrm{e}^x+\mathrm{e}^{-x})$$

and the odd part of e^x is

$$\frac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x})$$

We give these new functions special names: $\cosh x$ (pronounced 'cosh' x) and $\sinh x$ (pronounced 'shine' x).

Key Point 3

Hyperbolic Functions

$$\cosh x \equiv \frac{1}{2}(\mathrm{e}^x+\mathrm{e}^{-x})$$

$$\sinh x \equiv \frac{1}{2}(\mathrm{e}^x-\mathrm{e}^{-x})$$

These two functions, when added and subtracted, give

$$\cosh x+\sinh x \equiv \mathrm{e}^x \qquad \text{and} \qquad \cosh x-\sinh x \equiv \mathrm{e}^{-x}$$

The graphs of $\cosh x$ and $\sinh x$ are shown in Figure 4.

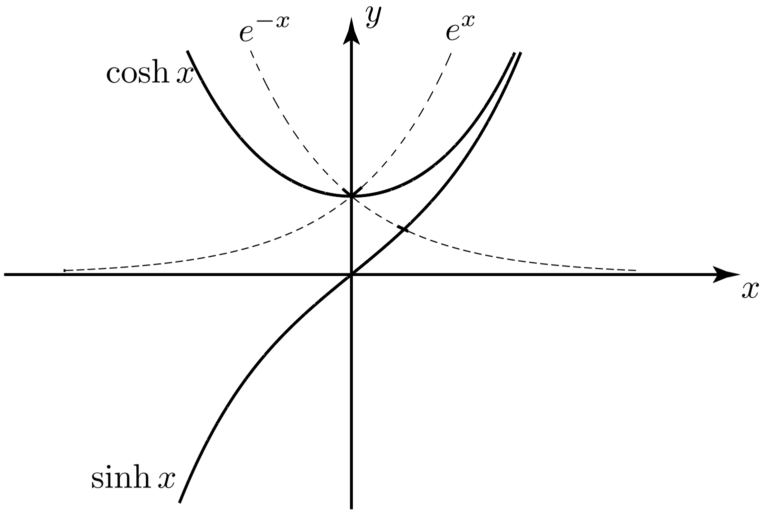


Figure 4: $\sinh x$ and $\cosh x$

Note that $\cosh x > 0$ for all values of x and that $\sinh x$ is zero only when $x = 0$.

2. Hyperbolic identities

The hyperbolic functions $\cosh x, \ \sinh x$ satisfy similar (but not exactly equivalent) identities to those satisfied by $\cos x, \ \sin x$. We note first some basic notation similar to that employed with trigonometric functions:

$$\cosh^n x \quad \text{means} \quad (\cosh x)^n \qquad\qquad \sinh^n x \quad \text{means} \quad (\sinh x)^n \qquad n \neq -1$$

In the special case that $n = -1$ we **do not** use $\cosh^{-1} x$ and $\sinh^{-1} x$ to mean $\dfrac{1}{\cosh x}$ and $\dfrac{1}{\sinh x}$ respectively. The notation $\cosh^{-1} x$ and $\sinh^{-1} x$ is reserved for the **inverse functions** of $\cosh x$ and $\sinh x$ respectively.

Task

Show that $\cosh^2 x - \sinh^2 x \equiv 1$ for all x.

(a) First, express $\cosh^2 x$ in terms of the exponential functions $\mathrm{e}^x, \ \mathrm{e}^{-x}$:

Your solution

$$\cosh^2 x \equiv \left[\frac{1}{2}(\mathrm{e}^x + \mathrm{e}^{-x})\right]^2 \equiv$$

Answer

$$\frac{1}{4}(\mathrm{e}^x + \mathrm{e}^{-x})^2 \equiv \frac{1}{4}[(\mathrm{e}^x)^2 + 2\mathrm{e}^x\mathrm{e}^{-x} + (\mathrm{e}^{-x})^2] \equiv \frac{1}{4}[\mathrm{e}^{2x} + 2\mathrm{e}^{x-x} + \mathrm{e}^{-2x}] \equiv \frac{1}{4}[\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}]$$

(b) Similarly, express $\sinh^2 x$ in terms of e^x and e^{-x}:

Your solution

$$\sinh^2 x \equiv \left[\frac{1}{2}(\mathrm{e}^x - \mathrm{e}^{-x})\right]^2 \equiv$$

Answer

$$\frac{1}{4}(\mathrm{e}^x - \mathrm{e}^{-x})^2 \equiv \frac{1}{4}[(\mathrm{e}^x)^2 - 2\mathrm{e}^x\mathrm{e}^{-x} + (\mathrm{e}^{-x})^2] \equiv \frac{1}{4}[\mathrm{e}^{2x} - 2\mathrm{e}^{x-x} + \mathrm{e}^{-2x}] \equiv \frac{1}{4}[\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}]$$

(c) Finally determine $\cosh^2 x - \sinh^2 x$ using the results from (a) and (b):

Your solution

$$\cosh^2 x - \sinh^2 x \equiv$$

Answer

$$\cosh^2 x - \sinh^2 x \equiv \frac{1}{4}[\mathrm{e}^{2x} + 2 + \mathrm{e}^{-2x}] - \frac{1}{4}[\mathrm{e}^{2x} - 2 + \mathrm{e}^{-2x}] \equiv 1$$

As an alternative to the calculation in this Task we could, instead, use the relations

$$\mathrm{e}^x \equiv \cosh x + \sinh x \qquad \mathrm{e}^{-x} \equiv \cosh x - \sinh x$$

and remembering the algebraic identity $(a+b)(a-b) \equiv a^2 - b^2$, we see that

$$(\cosh x + \sinh x)(\cosh x - \sinh x) \equiv \mathrm{e}^x\mathrm{e}^{-x} \equiv 1 \qquad \text{that is} \qquad \cosh^2 x - \sinh^2 x \equiv 1$$

Key Point 4

The fundamental identity relating hyperbolic functions is:

$$\cosh^2 x - \sinh^2 x \equiv 1$$

This is the hyperbolic function equivalent of the trigonometric identity: $\cos^2 x + \sin^2 x \equiv 1$

Task

Show that $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$.

First, express $\cosh x \cosh y$ in terms of exponentials:

Your solution

$$\cosh x \cosh y \equiv \left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\right) \equiv$$

Answer

$$\left(\frac{\mathrm{e}^{x}+\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}+\mathrm{e}^{-y}}{2}\right) \equiv \frac{1}{4}[\mathrm{e}^{x}\mathrm{e}^{y}+\mathrm{e}^{-x}\mathrm{e}^{y}+\mathrm{e}^{x}\mathrm{e}^{-y}+\mathrm{e}^{-x}\mathrm{e}^{-y}] \equiv \frac{1}{4}(\mathrm{e}^{x+y}+\mathrm{e}^{-x+y}+\mathrm{e}^{x-y}+\mathrm{e}^{-x-y})$$

Now express $\sinh x \sinh y$ in terms of exponentials:

Your solution

$$\left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2}\right) \equiv$$

Answer

$$\left(\frac{\mathrm{e}^{x}-\mathrm{e}^{-x}}{2}\right)\left(\frac{\mathrm{e}^{y}-\mathrm{e}^{-y}}{2}\right) \equiv \frac{1}{4}(\mathrm{e}^{x+y}-\mathrm{e}^{-x+y}-\mathrm{e}^{x-y}+\mathrm{e}^{-x-y})$$

Now express $\cosh x \cosh y + \sinh x \sinh y$ in terms of a hyperbolic function:

Your solution

$$\cosh x \cosh y + \sinh x \sinh y =$$

Answer

$\cosh x \cosh y + \sinh x \sinh y \equiv \frac{1}{2}(\mathrm{e}^{x+y}+\mathrm{e}^{-(x+y)})$ which we recognise as $\cosh(x+y)$

Other hyperbolic function identities can be found in a similar way. The most commonly used are listed in the following Key Point.

Key Point 5

Hyperbolic Identities

- $\cosh^2 - \sinh^2 \equiv 1$
- $\cosh(x+y) \equiv \cosh x \cosh y + \sinh x \sinh y$
- $\sinh(x+y) \equiv \sinh x \cosh y + \cosh x \sinh y$
- $\sinh 2x \equiv 2 \sinh x \cosh y$
- $\cosh 2x \equiv \cosh^2 x + \sinh^2 x$ or $\cosh 2x \equiv 2\cosh^2 - 1$ or $\cosh 2x \equiv 1 + 2\sinh^2 x$

3. Related hyperbolic functions

Given the trigonometric functions $\cos x$, $\sin x$ related functions can be defined; $\tan x$, $\sec x$, $\text{cosec } x$ through the relations:

$$\tan x \equiv \frac{\sin x}{\cos x} \qquad \sec x \equiv \frac{1}{\cos x} \qquad \text{cosec } x \equiv \frac{1}{\sin x} \qquad \cot x \equiv \frac{\cos x}{\sin x}$$

In an analogous way, given $\cosh x$ and $\sinh x$ we can introduce hyperbolic functions $\tanh x$, $\sec\text{h } x$, $\text{cosech } x$ and $\coth x$. These functions are defined in the following Key Point:

Key Point 6

Further Hyperbolic Functions

$$\begin{aligned} \tanh x &\equiv \frac{\sinh x}{\cosh x} \\ \text{sech } x &\equiv \frac{1}{\cosh x} \\ \text{cosech } x &\equiv \frac{1}{\sinh x} \\ \coth x &\equiv \frac{\cosh x}{\sinh x} \end{aligned}$$

Task Show that $1 - \tanh^2 x \equiv \text{sech}^2 x$

Use the identity $\cosh^2 x - \sinh^2 x \equiv 1$:

Your solution

Answer

Dividing both sides by $\cosh^2 x$ gives

$$1 - \frac{\sinh^2 x}{\cosh^2 x} \equiv \frac{1}{\cosh^2 x} \qquad \text{implying (see Key Point 6)} \qquad 1 - \tanh^2 x \equiv \text{sech}^2 x$$

Exercises

1. Express

   (a) $2\sinh x + 3\cosh x$ in terms of e^x and e^{-x}.

   (b) $2\sinh 4x - 7\cosh 4x$ in terms of e^{4x} and e^{-4x}.

2. Express

   (a) $2\text{e}^x - \text{e}^{-x}$ in terms of $\sinh x$ and $\cosh x$.

   (b) $\dfrac{7\text{e}^x}{(\text{e}^x - \text{e}^{-x})}$ in terms of $\sinh x$ and $\cosh x$, and then in terms of $\coth x$.

   (c) $4\text{e}^{-3x} - 3\text{e}^{3x}$ in terms of $\sinh 3x$ and $\cosh 3x$.

3. Using only the $\cosh$ and $\sinh$ keys on your calculator (or e^x key) find the values of

   (a) $\tanh 0.35$, (b) $\text{cosech } 2$, (c) $\text{sech } 0.6$.

Answers

1. (a) $\dfrac{5}{2}\text{e}^x - \dfrac{1}{2}\text{e}^{-x}$ (b) $-\dfrac{5}{2}\text{e}^{4x} - \dfrac{9}{2}\text{e}^{-4x}$

2. (a) $\cosh x + 3\sinh x$, (b) $\dfrac{7(\cosh x + \sinh x)}{2\sinh x}$, $\dfrac{7}{2}(\coth x + 1)$ (c) $\cosh 3x - 7\sinh 3x$

3. (a) 0.3364, (b) 0.2757 (c) 0.8436

Logarithms

6.3

Introduction

In this Section we introduce the logarithm: $\log_a b$. The operation of taking a logarithm essentially reverses the operation of raising a number to a power. We will formulate the basic laws satisfied by all logarithms and learn how to manipulate expressions involving logarithms. We shall see that to every law of indices there is an equivalent law of logarithms. Although logarithms to any positive base are defined it is common practice to employ only two kinds of logarithms: logs to base 10 and logs to base e.

Prerequisites

Before starting this Section you should ...

- have a knowledge of exponents and of the laws of indices

Learning Outcomes

On completion you should be able to ...

- invert $b = a^n$ using logarithms
- simplify expressions involving logarithms
- change bases in logarithms

1. Logarithms

Logarithms reverse the process of raising a base 'a' to a power 'n'. As with all exponentials, the base should be a positive number.

If $b = a^n$ then we write $\log_a b = n$.

Of course, the reverse statement is equivalent

If $\log_a b = n$ then $b = a^n$

The expression $\log_a b = n$ is read

"The log to base a of the number b is equal to n"

The term "log" is short for the word **logarithm**.

Example 3

Determine the log equivalents of
(a) $16 = 2^4$, (b) $16 = 4^2$, (c) $1000 = 10^3$,
(d) $134.896 = 10^{2.13}$, (e) $8.414867 = e^{2.13}$

Solution

(a) Since $16 = 2^4$ then $\log_2 16 = 4$
(b) Since $16 = 4^2$ then $\log_4 16 = 2$
(c) Since $1000 = 10^3$ then $\log_{10} 1000 = 3$
(d) Since $134.896 = 10^{2.13}$ then $\log_{10} 134.896 = 2.13$
(e) Since $8.41467 = \mathrm{e}^{2.13}$ then $\log_{\mathrm{e}} 8.414867 = 2.13$

Key Point 7

If	$b = a^n$	then	$\log_a b = n$
If	$\log_a b = n$	then	$b = a^n$

Task

Find the log equivalent of (a) $100 = 10^2$ (b) $\dfrac{1}{1000} = 10^{-3}$

Here, on the right-hand sides, the base is 10 in each case so:

Your solution

(a) $100 = 10^2$ implies

(b) $\dfrac{1}{1000} = 10^{-3}$ implies

Answer

(a) $\log_{10} 100 = 2$

(b) $\log_{10} \dfrac{1}{1000} = -3$

Task

Find the log equivalent of (a) $b = a^n$, (b) $c = a^m$, (c) $bc = a^{n+m}$

(a) Here the base is a so:

Your solution

$b = a^n$ implies $n =$

Answer

$n = \log_a b$

(b) Here the base is a so:

Your solution

$c = a^m$ implies $m =$

Answer

$m = \log_a c$

(c) Here the base is a so:

Your solution

$bc = a^{n+m}$ implies $n + m =$

Answer

$n + m = \log_a(bc)$

From the last Task we have found, using the property of indices, that

$$\log_a(bc) = n + m = \log_a b + \log_a c.$$

We conclude that the index law $a^n a^m = a^{n+m}$ has an equivalent logarithm law

$$\log_a(bc) = \log_a b + \log_a c$$

In words: "The log of a product is the sum of logs."

Indeed this property is one of the major advantages of using logarithms. They transform a **product** of numbers (a relatively difficult operation) to a **sum** of numbers (a relatively easy operation). Each index law has an equivalent logarithm law, true for any base, listed in the following Key Point:

Key Point 8

The laws of logarithms	**The laws of indices**
1. $\log_a(AB) = \log_a A + \log_a B$	1. $a^A a^B = a^{A+B}$
2. $\log_a(\dfrac{A}{B}) = \log_a A - \log_a B$	2. $a^A/a^B = a^{A-B}$
3. $\log_a(A^k) = k\log_a A$	3. $(a^A)^k = a^{kA}$
4. $\log_a(a^A) = A$	4. $a^{\log_a A} = A$
5. $\log_a a = 1$	5. $a^1 = a$
6. $\log_a 1 = 0$	6. $a^0 = 1$

2. Simplifying expressions involving logarithms

To simplify an expression involving logarithms their laws, given in Key Point 8, need to be used.

Example 4

Simplify: $\log_{10} 2 - \log_{10} 4 + \log_{10}(4^2) + \log_{10}(\dfrac{10}{4})$

Solution

The third term $\log_{10}(4^2)$ simplifies to $2\log_{10} 4$ and the last term

$$\log_{10}(\frac{10}{4}) = \log_{10} 10 - \log_{10} 4 = 1 - \log_{10} 4$$

So $\log_{10} 2 - \log_{10} 4 + \log_{10}(4^2) + \log_{10}(\dfrac{10}{4}) = \log_{10} 2 - \log_{10} 4 + 2\log_{10} 4 + 1 - \log_{10} 4 = \log_{10} 2 + 1$

Task

Simplify the expression:

$$\log_{10}(\frac{1}{10}) - \log_{10}(\frac{10}{27}) + \log_{10} 1000$$

(a) First simplify $\log_{10}(\frac{1}{10})$:

Your solution

$\log_{10}(\frac{1}{10}) =$

Answer

$\log_{10}(\frac{1}{10}) = \log_{10} 1 - \log_{10} 10 = 0 - 1 = -1$

(b) Now simplify $\log_{10}(\frac{10}{27})$:

Your solution

$\log_{10}(\frac{10}{27}) =$

Answer

$\log_{10}(\frac{10}{27}) = \log_{10} 10 - \log_{10} 27 = 1 - \log_{10} 27$

(c) Now simplify $\log_{10} 1000$:

Your solution

Answer

3

(d) Finally collect all the terms together from (a), (b), (c) and simplify:

Your solution

Answer

$-1 - (1 - \log_{10} 27) + 3 = 1 + \log_{10} 27$

3. Logs to base 10 and natural logs

In practice only two kinds of logarithms are commonly used, those to base 10, written $\log_{10}$ (or just simply $\log$) and those to base e, written $\log_e$ or more usually $\ln$ (called **natural logarithms**). Most scientific calculators will determine the logarithm to base 10 and to base e. For example,

$\log 13 = 1.11394$ (implying $10^{1.11394} = 13$), $\ln 23 = 3.13549$ (implying $e^{3.13549} = 23$)

Task

Use your calculator to determine (a) $\log 10$, (b) $\log 1000000$, (c) $\log 0.1$

Your solution

(a) $\log 10 =$ (b) $\log 1000000 =$ (c) $\log 0.1 =$

Answer

(a) 1, (b) 6, (c) -1.

Each of the above results could be determined directly, without the use of a calculator. For example:
Since $\log_a a = 1$ then $\log 10\ (\equiv \log_{10} 10) = 1$.
Since $\log_a A^k = k \log_a A$ then $\log 1000000 = \log 10^6 = 6 \log 10 = 6$.

Since $\log_a(\dfrac{A}{B}) = \log_a A - \log_a B$ and $\log_a 1 = 0$ and $\log_a a = 0$, then

$\log 0.1 = \log(\dfrac{1}{10}) = \log 1 - \log(10) = -1$

Task

Use your calculator to determine
(a) $\ln 29.42$, (b) $\ln \mathrm{e}$, (c) $\ln 0.1$

Your solution

(a) $\ln 29.42 =$ (b) $\ln \mathrm{e} =$ (c) $\ln 0.1 =$

Answer

(a) $\ln 29.42 = 3.38167$, (b) $\ln \mathrm{e} = 1$, (c) $\ln 0.1 = -2.30258$

4. Changing base in logarithms

It is sometimes required to express the logarithm with respect to one base in terms of a logarithm with respect to another base.
Now

$$b = a^n \qquad \text{implies} \qquad \log_a b = n$$

where we have used logs to base a. What happens if, for some reason, we want to use another base, p say? We take logs (to base p) of both sides of $b = a^n$:

$$\log_p(b) = \log_p(a^n) = n \log_p a \qquad \text{(using one of the logarithm laws)}$$

So

$$n = \frac{\log_p(b)}{\log_p(a)} \qquad \text{that is} \qquad \log_a b = \frac{\log_p(b)}{\log_p(a)}$$

This is the rule to be used when converting logarithms from one base to another.

Key Point 9

$$\log_a b = \frac{\log_p b}{\log_p a}$$

For base 10 logs:

$$\log_a b = \frac{\log(b)}{\log(a)}$$

For example,

$$\log_3 7 = \frac{\log 7}{\log 3} = \frac{0.8450980}{0.4771212} = 1.7712437$$

(Check, on your calculator, that $3^{1.7712437} = 7$).
For natural logs:

$$\log_a b = \frac{\ln(b)}{\ln(a)}$$

For example,

$$\log_3 7 = \frac{\ln 7}{\ln 3} = \frac{1.9459101}{1.0986123} = 1.7712437$$

Of course, $\log_3 7$ cannot be determined directly on your calculator since logs to base 3 are not available but it can be found using the above method.

Task

Use your calculator to determine the value of $\log_{21} 7$ using first base 10 then check using base e.

Re-express $\log_{21} 7$ using base 10 then base e:

Your solution

$$\log_{21} 7 = \frac{\log 7}{\log 21} = \qquad\qquad \log_{21} 7 = \frac{\ln 7}{\ln 21} =$$

Answer

$$\log_{21} 7 = \frac{\log 7}{\log 21} = 0.6391511 \qquad\qquad \log_{21} 7 = \frac{\ln 7}{\ln 21} = 0.6391511$$

Example 5

Simplify the expression $10^{\log x}$.

Solution

Let $y = 10^{\log x}$ then take logs (to base 10) of both sides:

$$\log y = \log(10^{\log x}) = (\log x)\log 10$$

where we have used: $\log A^k = k\log A$. However, since we are using logs to base 10 then $\log 10 = 1$ and so

$$\log y = \log x \qquad \text{implying} \qquad y = x$$

Therefore, finally we conclude that

$$10^{\log x} = x$$

This is an important result true for logarithms of any base. It follows from the basic definition of the logarithm.

Key Point 10

$$a^{\log_a x} = x$$

Raising to the power and taking logs are **inverse** operations.

Exercises

1. Find the values of (a) $\log_2 8$ (b) $\log_{16} 50$ (c) $\ln 28$

2. Simplify

   (a) $\log 1 - 3\log 2 + \log 16$.

   (b) $10\log x - 2\log x^2$.

   (c) $\ln(8x-4) - \ln(4x-2)$.

   (d) $\ln 10 \log 7 - \ln 7$.

Answers

1. (a) 3 (b) 1.41096 (c) 3.3322
2. (a) $\log 2$, (b) $6\log x$ or $\log x^6$, (c) $\ln 2$, (d) 0

The Logarithmic Function

6.4

Introduction

In this Section we consider the logarithmic function $y = \log_a x$ and examine its important characteristics. We see that this function is only defined if x is a positive number. We also see that the $\log$ function is the inverse of the exponential function and vice versa. We show, through numerous examples, how equations involving logarithms and exponentials can be solved.

Prerequisites

Before starting this Section you should ...

- have knowledge of inverse functions
- have knowledge of the laws of logarithms and of the laws of indices
- be able to solve quadratic equations

Learning Outcomes

On completion you should be able to ...

- explain the relation between the logarithm and the exponential function
- solve equations involving exponentials and logarithms

1. The logarithmic function

In Section 6.3 we introduced the operation of taking logarithms which reverses the operation of exponentiation.

If $a > 0$ and $a \neq 1$ then $\quad x = a^y \quad$ implies $\quad y = \log_a x$

In this Section we consider the log function in more detail. We shall concentrate only on the functions $\log x$ (i.e. to base 10) and $\ln x$ (i.e. to base e). The functions $y = \log x$ and $y = \ln x$ have similar characteristics. We can never choose x as a negative number since 10^y and e^y are each always positive. The graphs of $y = \log x$ and $y = \ln x$ are shown in Figure 5.

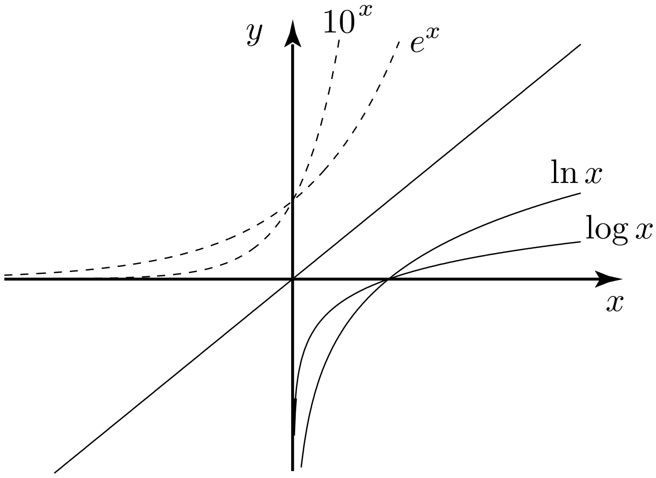


Figure 5: Logarithmic and exponential functions

From the graphs we see that both functions are one-to-one so each has an **inverse function** - the inverse function of $\log_a x$ is a^x. Let us do this for logs to base 10.

2. Solving equations involving logarithms and exponentials

To solve equations which involve logarithms or exponentials we need to be aware of the basic laws which govern both of these mathematical concepts. We illustrate by considering some examples.

Example 6

Solve for the variable x: (a) $3 = 10^x$, (b) $10^{x/4} = \log 3$, (c) $\dfrac{1}{17 - e^x} = 4$

Solution

(a) Here we take logs (to base 10 because of the term 10^x) of both sides to get

$$\log 3 = \log 10^x = x \log 10 = x$$

where we have used the general property that $\log_a A^k = k \log_a A$ and the specific property that $\log 10 = 1$. Hence $x = \log 3$ or, in numerical form, $x = 0.47712$ to 5 d.p.

Solution (contd.)

(b) The approach used in (a) is used here. Take logs of both sides: $\log(10^{x/4}) = \log(\log 3)$

that is $\quad \dfrac{x}{4}\log 10 = \log(\log 3) = \log(0.4771212) = -0.3213712$

So, since $\log 10 = 1$, we have $\quad x = 4(-0.3213712) = -1.28549 \quad$ to 5 d.p.

(c) Here we simplify the expression before taking logs.

$$\frac{1}{17 - \mathrm{e}^x} = 4 \qquad \text{implies} \qquad 1 = 4(17 - \mathrm{e}^x)$$

or $\;4\mathrm{e}^x = 4(17) - 1 = 67\;$ so $\;\mathrm{e}^x = 16.75$. Now taking natural logs of both sides (because of the presence of the e^x term) we have:

$$\ln(\mathrm{e}^x) = \ln(16.75) = 2.8183983$$

But $\ln(\mathrm{e}^x) = x\ln \mathrm{e} = x$ and so the solution to $\dfrac{1}{17 - \mathrm{e}^x} = 4$ is $x = 2.81840$ to 5 d.p.

Task

Solve the equation $(\mathrm{e}^x)^2 = 50$

First solve for e^x by taking square roots of both sides:

Your solution

$(\mathrm{e}^x)^2 = 50 \quad$ implies $\quad \mathrm{e}^x =$

Answer

$(\mathrm{e}^x)^2 = 50 \quad$ implies $\quad \mathrm{e}^x = \sqrt{50} = 7.071068$. Here we have taken the positive value for the square root since we know that exponential functions **are always positive**.

Now take logarithms to an appropriate base to find x:

Your solution

$\mathrm{e}^x = 7.071068 \quad$ implies $\quad x =$

Answer

$\mathrm{e}^x = 7.071068 \quad$ implies $\quad x = \ln(7.071068) = 1.95601$ to 5 d.p.

Solve the equation $e^{2x} = 17e^x$

First simplify the expression as much as possible (divide both sides by e^x):

Your solution

$$e^{2x} = 17e^x \quad \text{implies} \quad \frac{e^{2x}}{e^x} = 17 \quad \text{so}$$

Answer

$\dfrac{e^{2x}}{e^x} = 17$ implies $e^{2x-x} = 17$ so $e^x = 17$

Now complete the solution for x:

Your solution

$$e^x = 17 \quad \text{implies} \quad x =$$

Answer

$x = \ln(17) = 2.8332133$

Example 7

Find x if $\quad 10^x - 5 + 6(10^{-x}) = 0$

Solution

We first simplify this expression by multiplying through by 10^x (to eliminate the term 10^{-x}):

$$10^x(10^x) - 10^x(5) + 10^x(6(10^{-x})) = 0$$

or

$$(10^x)^2 - 5(10^x) + 6 = 0 \qquad \text{since} \qquad 10^x(10^{-x}) = 10^0 = 1$$

We realise that this expression is a **quadratic equation**. Let us put $y = 10^x$ to give

$$y^2 - 5y + 6 = 0$$

Now, we can factorise to give

$$(y-3)(y-2) = 0 \qquad \text{so that} \qquad y = 3 \quad \text{or} \quad y = 2$$

For each of these values of y we obtain a separate value for x since $y = 10^x$.

Case 1 If $y = 3$ then $3 = 10^x$ implying $x = \log 3 = 0.4771212$

Case 2 If $y = 2$ then $2 = 10^x$ implying $x = \log 2 = 0.3010300$

We conclude that the equation $10^x - 5 + 6(10^{-x}) = 0$ has two possible solutions for x: either $x = 0.4771212$ or $x = 0.3010300$, to 7 d.p.

Task

Solve $2e^{2x} - 7e^{x} + 3 = 0.$

First write this equation as a quadratic in the variable $y = e^{x}$ remembering that $e^{2x} \equiv (e^{x})^2$:

Your solution

If $y = e^{x}$ then $2e^{2x} - 7e^{x} + 3 = 0$ becomes

Answer

$2y^2 - 7y + 3 = 0$

Now solve the quadratic for y:

Your solution

$2y^2 - 7y + 3 = 0$ implies $(2y \qquad)(y \qquad) = 0$

Answer

$(2y - 1)(y - 3) = 0$ therefore $y = \dfrac{1}{2}$ or $y = 3$

Finally, for each of your values of y, find x:

Your solution

If $y = \dfrac{1}{2}$ then $\dfrac{1}{2} = e^{x}$ implies $x =$

If $y = 3$ then $3 = e^{x}$ implies $x =$

Answer

$x = -0.693147$ or $x = 1.0986123$

Task

The temperature T , in degrees C, of a chemical reaction is given by the formula

$T = 80e^{0.03t} \times \ t \geq 0$, where t is the time, in seconds.

Calculate the time taken for the temperature to reach 150° C .

Answer

$$150 = 80e^{0.03t} \quad \Rightarrow \quad 1.875 = e^{0.03t} \quad \Rightarrow \quad \ln(1.875) = 0.03t \quad \Rightarrow \quad t = \frac{\ln(1.875)}{0.03}$$

This gives $t = 20.95$ to 2 d.p.

So the time is 21 seconds.

Engineering Example 1

Arrhenius' law

Introduction

Chemical reactions are very sensitive to temperature; normally, the rate of reaction increases as temperature increases. For example, the corrosion of iron and the spoiling of food are more rapid at higher temperatures. Chemically, the probability of collision between two molecules increases with temperature, and an increased collision rate results in higher kinetic energy, thus increasing the proportion of molecules that have the **activation energy** for the reaction, i.e. the minimum energy required for a reaction to occur. Based upon his observations, the Swedish chemist, Svante Arrhenius, proposed that the rate of a chemical reaction increases exponentially with temperature. This relationship, now known as Arrhenius' law, is written as

$$k = k_0 \exp\left(\frac{-E_a}{RT}\right) \qquad (1)$$

where k is the reaction rate constant, k_0 is the frequency factor, E_a is the activation energy, R is the universal gas constant and T is the absolute temperature. Thus, the reaction rate constant, k, depends on the quantities k_0 and E_a, which characterise a given reaction, and are generally assumed to be temperature independent.

Problem in words

In a laboratory, ethyl acetate is reacted with sodium hydroxide to investigate the reaction kinetics. Calculate the frequency factor and activation energy of the reaction from Arrhenius' Law, using the experimental measurements of temperature and reaction rate constant in the table:

T	310	350
k	7.757192	110.9601

Mathematical statement of problem

Given that $k = 7.757192\ \text{s}^{-1}$ at $T = 310$ K and $k = 110.9601\ \text{s}^{-1}$ at $T = 350$ K, use Equation (1) to produce two linear equations in E_a and k_0. Solve these to find E_a and k_0. (Assume that the gas constant $R = 8.314\ \text{J K}^{-1}\ \text{mol}^{-1}$.)

Mathematical analysis

Taking the natural logarithm of both sides of (1)

$$\ln k = \ln\left\{k_0 \exp\left(\frac{-E_a}{RT}\right)\right\} = \ln k_0 - \frac{E_a}{RT}$$

Now inserting the experimental data gives the two linear equations in E_a and k_0

$$\ln k_1 = \ln k_0 - \frac{E_a}{RT_1} \qquad (2)$$

$$\ln k_2 = \ln k_0 - \frac{E_a}{RT_2} \qquad (3)$$

where $k_1 = 7.757192$, $T_1 = 310$ and $k_2 = 110.9601$, $T_2 = 350$.

Firstly, to find E_a, subtract Equation (2) from Equation (3)

$$\ln k_2 - \ln k_1 = \frac{E_a}{RT_1} - \frac{E_a}{RT_2} = \frac{E_a}{R}\left(\frac{1}{T_1} - \frac{1}{T_2}\right)$$

so that

$$E_a = \frac{R\,(\ln k_2 - \ln k_1)}{\left(\dfrac{1}{T_1} - \dfrac{1}{T_2}\right)}$$

and substituting the values gives

$$E_a = 60000 \text{ J mol}^{-1} = 60 \text{ kJ mol}^{-1}$$

Secondly, to find k_0, from (2)

$$\ln k_0 = \ln k_1 + \frac{E_a}{RT_1} \qquad \Rightarrow \qquad k_0 = \exp\left(\ln k_1 + \frac{E_a}{RT_1}\right) = k_1 \exp\left(\frac{E_a}{RT_1}\right)$$

and substituting the values gives

$$k_0 = 1.0 \times 10^{11} \text{ s}^{-1}$$

Task

The reaction

$$2NO_2(g) \longrightarrow 2NO(g) + O_2(g)$$

has a reaction rate constant of 1.0×10^{-10} s^{-1} at 300 K and activation energy of 111 kJ mol^{-1} = 111 000 J mol^{-1}. Use Arrhenius' law to find the reaction rate constant at a temperature of 273 K.

Your solution

Answer

Rearranging Arrhenius' equation gives

$$k_0 = k\,\exp\left(\frac{E_a}{RT}\right)$$

Substituting the values gives $k_0 = 2.126 \times 10^9$ s^{-1}

Now we use this value of k_0 with E_a in Arrhenius' equation (1) to find k at $T = 273$ K

$$k = k_0\,\exp\left(\frac{-E_a}{RT}\right) = 1.226 \times 10^{-12} \text{ s}^{-1}$$

Task

For a chemical reaction with frequency factor $k_0 = 0.5$ s^{-1} and ratio $E_a/R = 800$ K, use Arrhenius' law to find the temperature at which the reaction rate constant would be equal to 0.1 s^{-1}.

Your solution

Answer

Rearranging Equation (1)

$$\frac{k}{k_0} = \exp\left(\frac{-E_a}{RT}\right)$$

Taking the natural logarithm of both sides

$$\ln\left(\frac{k}{k_0}\right) = \frac{-E_a}{RT}$$

so that

$$T = \frac{-E_a}{R\,\ln(k/k_0)} = \frac{E_a}{R\,\ln(k_0/k)}$$

Substituting the values gives $T = 497$ K

As a final example we consider equations involving the hyperbolic functions.

Example 8

Solve the equations

(a) $\cosh 3x = 1$ (b) $\cosh 3x = 2$ (c) $2\cosh^2 x = 3\cosh 2x - 3$

Solution

(a) From its graph we know that $\cosh x = 0$ only when $x = 0$, so we need $3x = 0$ which implies $x = 0$.

(b) $\cosh 3x = 2$ implies $\dfrac{e^{3x} + e^{-3x}}{2} = 2$ or $e^{3x} + e^{-3x} - 4 = 0$

Now multiply through by e^{3x} (to eliminate the term e^{-3x}) to give

$$e^{3x}e^{3x} + e^{3x}e^{-3x} - 4e^{3x} = 0 \quad \text{or} \quad (e^{3x})^2 - 4e^{3x} + 1 = 0$$

This is a quadratic equation in the variable e^{3x} so substituting $y = e^{3x}$ gives

$$y^2 - 4y + 1 = 0 \quad \text{implying} \quad y = 2 \pm \sqrt{3} \quad \text{so} \quad y = 3.7321 \quad \text{or} \quad 0.26795$$

$$e^{3x} = 3.7321 \quad \text{implies} \quad x = \frac{1}{3}\ln 3.7321 = 0.439 \quad \text{to 3 d.p.}$$

$$e^{3x} = 0.26795 \quad \text{implies} \quad x = \frac{1}{3}\ln 0.26795 = -0.439 \quad \text{to 3 d.p.}$$

(c) We first simplify this expression by using the identity: $\cosh 2x = 2\cosh^2 - 1$. Thus the original equation $2\cosh^2 x = 3\cosh 2x - 3$ becomes $\cosh 2x + 1 = 3\cosh 2x - 3$ or, when written in terms of exponentials:

$$\frac{e^{2x} + e^{-2x}}{2} = 3\left(\frac{e^{2x} + e^{-2x}}{2}\right) - 4$$

Multiplying through by $2e^{2x}$ gives $e^{4x} + 1 = 3(e^{4x} + 1) - 8e^{2x}$ or, after simplifying:

$$e^{4x} - 4e^{2x} + 1 = 0$$

Writing $y = e^{2x}$ we easily obtain $y^2 - 4y + 1 = 0$ with solution (using the quadratic formula):

$$y = \frac{4 \pm \sqrt{16 - 4}}{2} = 2 \pm \sqrt{3}$$

If $y = 2 + \sqrt{3}$ then $2 + \sqrt{3} = e^{2x}$ implying $x = 0.65848$ to 5 d.p.

If $y = 2 - \sqrt{3}$ then $2 - \sqrt{3} = e^{2x}$ implying $x = -0.65848$ to 5 d.p.

Find the solution for x if $\tanh x = 0.5$.

First re-write $\tanh x$ in terms of exponentials:

Your solution

$\tanh x =$

Answer

$$\tanh x = \frac{\mathrm{e}^x - \mathrm{e}^{-x}}{\mathrm{e}^x + \mathrm{e}^{-x}} = \frac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1}$$

Now substitute into $\tanh x = 0.5$:

Your solution

$\tanh x = 0.5$ implies $\dfrac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1} = 0.5$ so, on simplifying, $\mathrm{e}^{2x} =$

Answer

$\dfrac{\mathrm{e}^{2x} - 1}{\mathrm{e}^{2x} + 1} = 0.5$ implies $(\mathrm{e}^{2x} - 1) = \dfrac{1}{2}(\mathrm{e}^{2x} + 1)$ so $\dfrac{\mathrm{e}^{2x}}{2} = \dfrac{3}{2}$ so, finally, $\mathrm{e}^{2x} = 3$

Now complete your solution by finding x:

Your solution

$\mathrm{e}^{2x} = 3$ so $x =$

Answer

$$x = \frac{1}{2}\ln 3 = 0.549306$$

Alternatively, many calculators can directly calculate the inverse function $\tanh^{-1}$. If you have such a calculator then you can use the fact that

$\tanh x = 0.5$ implies $x = \tanh^{-1} 0.5$ to obtain directly $x = 0.549306$

Example 9

Solve for x if $3\ln x + 4\log x = 1$.

Solution

This has logs to two different bases. So we must first express each logarithm in terms of logs to the *same* base, e say. From Key Point 8

$$\log x = \frac{\ln x}{\ln 10}$$

So $3\ln x + 4\log x = 1$ becomes

$$3\ln x + 4\frac{\ln x}{\ln 10} = 1 \qquad \text{or} \qquad (3 + \frac{4}{\ln 10})\ln x = 1$$

leading to $\ln x = \dfrac{\ln 10}{3\ln 10 + 4} = \dfrac{2.302585}{10.907755} = 0.211096$ and so

$$x = \mathrm{e}^{0.211096} = 1.2350311$$

Exercises

1. Solve for the variable x: (a) $\pi = 10^x$ (b) $10^{-x/2} = 3$ (c) $\dfrac{1}{17 - \pi^x} = 4$

2. Solve the equations

   (a) $\mathrm{e}^{2x} = 17\mathrm{e}^x$, (b) $\mathrm{e}^{2x} - 2\mathrm{e}^x - 6 = 0$, (c) $\cosh x = 3$.

Answers

1. (a) $x = \log\pi = 0.497$

   (b) $-x/2 = \log 3$ and so $x = -2\log 3 = -0.954$

   (c) $17 - \pi^x = 0.25$ so $\pi^x = 16.75$ therefore $x = \dfrac{\log 16.75}{\log\pi} = \dfrac{1.224}{0.497} = 2.462$

2. (a) Take logs of both sides: $2x = \ln 17 + x \qquad \therefore \qquad x = \ln 17 = 2.833$

   (b) Let $y = \mathrm{e}^x$ then $y^2 - 2y - 6 = 0$ therefore $y = 1 \pm \sqrt{7}$ (we cannot take the negative sign since exponentials can never be negative). Thus $x = \ln(1 + \sqrt{7}) = 1.2936$.

   (c) $\mathrm{e}^x + \mathrm{e}^{-x} = 6$ therefore $\mathrm{e}^{2x} - 6\mathrm{e}^x + 1 = 0$ so $\mathrm{e}^x = \dfrac{6 \pm \sqrt{36 - 4}}{2} = 3 \pm \sqrt{8}$

   We have, finally $x = \ln(3 + \sqrt{8}) = 1.7627$ or $x = \ln(3 - \sqrt{8}) = -1.7627$

Modelling Exercises

6.5

Introduction

This Section provides examples and tasks employing exponential functions and logarithmic functions, such as growth and decay models which are important throughout science and engineering.

Prerequisites

Before starting this Section you should ...

- be familiar with the laws of logarithms
- have knowledge of logarithms to base 10
- be able to solve equations involving logarithms and exponentials

Learning Outcomes

On completion you should be able to ...

- develop exponential growth and decay models

1. Exponential increase

Task

(a) Look back at Section 6.2 to review the definitions of *an* exponential function and *the* exponential function.

(b) List examples in this Workbook of contexts in which exponential functions are appropriate.

Your solution

Answer

(a) *An* exponential function has the form $y = a^x$ where $a > 0$. *The* exponential function has the form $y = e^x$ where $e = 2.718282......$

(b) It is stated that exponential functions are useful when modelling the shape of a hanging chain or rope under the effect of gravity or for modelling exponential growth or decay.

We will look at a specific example of the exponential function used to model a population increase.

Task

Given that

$$P = 12e^{0.1t} \qquad (0 \leq t \leq 25)$$

where P is the number in the population of a city **in millions** at time t **in years** answer these questions.

(a) What does this model imply about P when $t = 0$?

(b) What is the stated upper limit of validity of the model?

(c) What does the model imply about values of P over time?

(d) What does the model predict for P when $t = 10$? Comment on this.

(d) What does the model predict for P when $t = 25$? Comment on this.

Your solution

(a)

(b)

(c)

(d)

(e)

Answer

(a) At $t = 0$, $P = 12$ which represents the initial population of 12 million. (Recall that $e^0 = 1$.)

(b) The time interval during which the model is valid is stated as $(0 \leq t \leq 25)$ so the model is intended to apply for 25 years.

(c) This is exponential growth so P will increase from 12 million at an accelerating rate.

(d) $P(10) = 12\text{e}^1 \approx 33$ million. This is getting very large for a city but might be attainable in 10 years and just about sustainable.

(e) $P(25) = 12\text{e}^{2.5} \approx 146$ million. This is unrealistic for a city.

Note that exponential population growth of the form $P = P_0 e^{kt}$ means that as t becomes large and positive, P becomes very large. Normally such a population model would be used to predict values of P for $t > 0$, where $t = 0$ represents the present or some fixed time when the population is known. In Figure 6, values of P are shown for $t < 0$. These correspond to extrapolation of the model into the past. Note that as t becomes increasingly negative, P becomes very small but is never zero or negative because e^{kt} is positive for all values of t. The parameter k is called the **instantaneous fractional growth rate**.

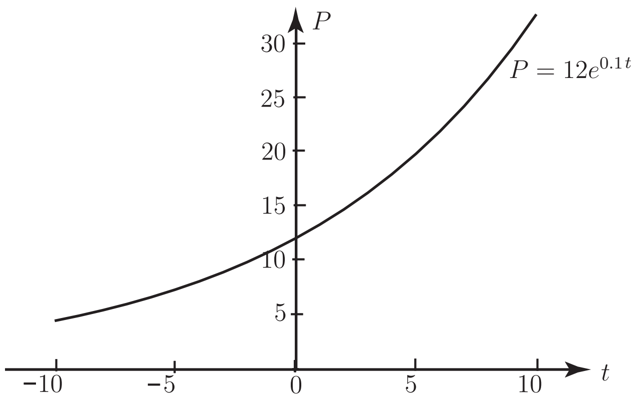


Figure 6: The function $P = 12e^{0.01t}$

For the model $P = 12e^{kt}$ we see that $k = 0.1$ is unrealistic, and more realistic values would be $k = 0.01$ or $k = 0.02$. These would be similar but k=0.02 implies a faster growth for $t > 0$ than $k = 0.01$. This is clear in the graphs for $k = 0.01$ and $k = 0.02$ in Figure 7. The functions are plotted up to 200 years to emphasize the increasing difference as t increases.

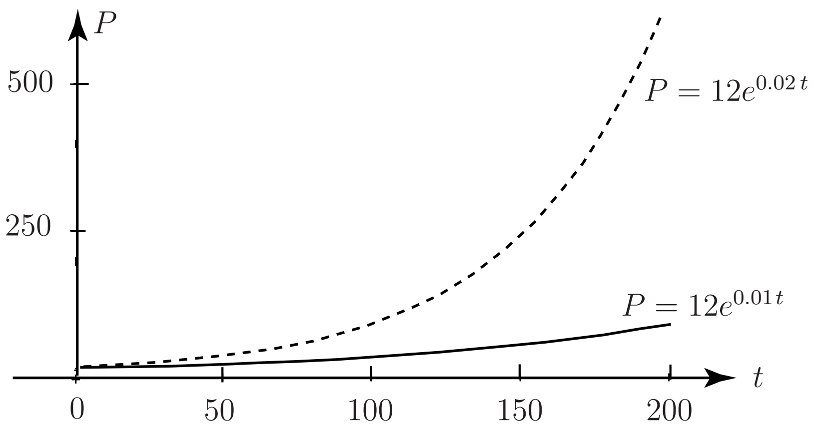


Figure 7: Comparison of the functions $P = 12e^{0.01t}$ and $P = 12e^{0.02t}$

The exponential function may be used in models for other types of growth as well as population growth. A general form may be written

$$y = ae^{bx} \qquad a > 0, \quad b > 0, \quad c \leq x \leq d$$

where a represents the value of y at $x = 0$. The value a is the intercept on the y-axis of a graphical representation of the function. The value b controls the rate of growth and c and d represent limits on x.

In the general form, a and b represent the **parameters** of the exponential function which can be selected to fit any given modelling situation where an exponential function is appropriate.

2. Linearisation of exponential functions

This subsection relates to the description of log-linear plots covered in Section 6.6.

Frequently in engineering, the question arises of how the parameters of an exponential function might be found from given data. The method follows from the fact that it is possible to 'undo' the exponential function and obtain a linear function by means of the logarithmic function. Before showing the implications of this method, it may be necessary to remind you of some rules for manipulating logarithms and exponentials. These are summarised in Table 1 on the next page, which exactly matches the general list provided in Key Point 8 in Section 6.3 (page 22.)

Table 1: Rules for manipulating base e logarithms and exponentials

Number	Rule	Number	Rule
1a	$\ln(xy) = \ln(x) + \ln(y)$	1b	$\mathrm{e}^x \times \mathrm{e}^y = \mathrm{e}^{x+y}$
2a	$\ln(x/y) = \ln(x) - \ln(y)$	2b	$\mathrm{e}^x / \mathrm{e}^y = \mathrm{e}^{x-y}$
3a	$\ln(x^y) = y\ln(x)$	3b	$(\mathrm{e}^x)^y = \mathrm{e}^{xy}$
4a	$\ln(\mathrm{e}^x) = x$	4b	$\mathrm{e}^{\ln(x)} = x$
5a	$\ln(\mathrm{e}) = 1$	5b	$\mathrm{e}^1 = \mathrm{e}$
6a	$\ln(1) = 0$	6b	$\mathrm{e}^0 = 1$

We will try 'undoing' the exponential in the particular example

$$P = 12e^{0.1t}$$

We take the natural logarithm ($\ln$) of both sides, which means logarithm to the base e. So

$$\ln(P) = \ln(12e^{0.1t})$$

The result of using Rule 1a in Table 1 is

$$\ln(P) = \ln(12) + \ln(e^{0.1t}).$$

The natural logarithmic functions 'undoes' the exponential function, so by Rule 4a,

$$\ln(e^{0.1t}) = 0.1t$$

and the original equation for P becomes

$$\ln(P) = \ln(12) + 0.1t.$$

Compare this with the general form of a linear function $y = ax + b$.

$$\begin{array}{ccccc} y & = & ax & + & b \\ \downarrow & & \downarrow & & \downarrow \\ \ln(P) & = & 0.1t & + & \ln(12) \end{array}$$

If we regard $\ln(P)$ as equivalent to y, 0.1 as equivalent to the constant a, t as equivalent to x, and $\ln(12)$ as equivalent to the constant b, then we can identify a linear relationship between $\ln(P)$ and t. A plot of $\ln(P)$ against t should result in a straight line, of slope 0.1, which crosses the $\ln(P)$ axis at $\ln(12)$. (Such a plot is called a **log-linear** or **log-lin** plot.) This is not particularly interesting here because we know the values 12 and 0.1 already.

Suppose, though, we want to try using the general form of the exponential function

$$P = ae^{bt} \qquad (c \leq t \leq d)$$

to create a continuous model for a population for which we have some discrete data. The first thing to do is to take logarithms of both sides

$$\ln(P) = \ln(ae^{bt}) \qquad (c \leq t \leq d).$$

Rule 1 from Table 1 then gives

$$\ln(P) = \ln(a) + \ln(e^{bt}) \qquad (c \leq t \leq d).$$

But, by Rule 4a, $\ln(e^{bt}) = bt$, so this means that

$$\ln(P) = \ln(a) + bt \qquad (c \leq t \leq d).$$

So, given some 'population versus time' data, for which you believe can be modelled by some version of the exponential function, plot the natural logarithm of population against time. If the exponential function is appropriate, the resulting data points should lie on or near a straight line. The slope of the straight line will give an estimate for b and the intercept with the $\ln(P)$ axis will give an estimate for $\ln(a)$. You will have carried out a **logarithmic transformation** of the original data for P. We say the original variation has been **linearised**.

A similar procedure will work also if any exponential function rather than the base e exponential function is used. For example, suppose that we try to use the function

$$P = A \times 2^{Bt} \qquad (C \leq t \leq D),$$

where A and B are constant parameters to be derived from the given data. We can take natural logarithms again to give

$$\ln(P) = \ln(A \times 2^{Bt}) \qquad (C \leq t \leq D).$$

Rule 1a from Table 1 then gives

$$\ln(P) = \ln(A) + \ln(2^{Bt}) \qquad (C \leq t \leq D).$$

Rule 3a then gives

$$\ln(2^{Bt}) = Bt\ln(2) = B\ln(2)\ t$$

and so

$$\ln(P) = \ln(A) + B\ln(2)\ t \qquad (C \leq t \leq D).$$

Again we have a straight line graph with the same intercept as before, $\ln A$, but this time with slope $B\ln(2)$.

Task

The amount of money $\pounds M$ to which $\pounds 1$ grows after earning interest of 5% p.a. for N years is worked out as

$$M = 1.05^N$$

Find a linearised form of this equation.

Your solution

Answer

Take natural logarithms of both sides.

$$\ln(M) = \ln(1.05^N).$$

Rule 3b gives

$$\ln(M) = N\ln(1.05).$$

So a plot of $\ln(M)$ against N would be a straight line passing through $(0, 0)$ with slope $\ln(1.05)$.

The linearisation procedure also works if logarithms other than natural logarithms are used. We start again with

$$P = A \times 2^{Bt} \qquad (C \leq t \leq D).$$

and will take logarithms to base 10 instead of natural logarithms. Table 2 presents the laws of logarithms and indices (based on Key Point 8 page 22) interpreted for $\log_{10}$.

Table 2: Rules for manipulating base 10 logarithms and exponentials

Number	Rule	Number	Rule
1a	$\log_{10}(AB) = \log_{10} A + \log_{10} B$	1b	$10^A 10^B = 10^{A+B}$
2a	$\log_{10}(A/B) = \log_{10} A - \log_{10} B$	2b	$10^A/10^B = 10^{A-B}$
3a	$\log_{10}(A^k) = k \log_{10} A$	3b	$(10^A)^k = 10^{kA}$
4a	$\log_{10}(10^A) = A$	4b	$10^{\log_{10} A} = A$
5a	$\log_{10} 10 = 1$	5b	$10^1 = 10$
6a	$\log_{10} 1 = 0$	6b	$10^0 = 1$

Taking logs of $P = A \times 2^{Bt}$ gives:

$$\log_{10}(P) = \log_{10}(A \times 2^{Bt}) \qquad (C \leq t \leq D).$$

Rule 1a from Table 2 then gives

$$\log_{10}(P) = \log_{10}(A) + \log_{10}(2^{Bt}) \qquad (C \leq t \leq D).$$

Use of Rule 3a gives the result

$$\log_{10}(P) = \log_{10}(A) + B \log_{10}(2)\ t \qquad (C \leq t \leq D).$$

Task

(a) Write down the straight line function corresponding to taking logarithms of the general exponential function

$$P = ae^{bt} \qquad (c \leq t \leq d)$$

by taking logarithms to base 10.

(b) Write down the slope of this line.

Your solution

Answer

(a) $\log_{10}(P) = \log_{10}(a) + (b \log_{10}(e))t \qquad (c \leq t \leq d)$

(b) $b \log_{10}(e)$

It is not usually necessary to declare the subscript 10 when indicating logarithms to base 10. If you meet the term 'log' it will probably imply "to the base 10". In the remainder of this Section, the subscript 10 is dropped where $\log_{10}$ is implied.

3. Exponential decrease

Consider the value, $£D$, of a car subject to depreciation, in terms of the age A years of the car. The car was bought for $£10500$. The function

$$D = 10500e^{-0.25A} \qquad (0 \leq A \leq 6)$$

could be considered appropriate on the ground that (a) D had a fixed value of $£10500$ when $A = 0$, (b) D decreases as A increases and (c) D decreases faster when A is small than when A is large. A plot of this function is shown in Figure 8.

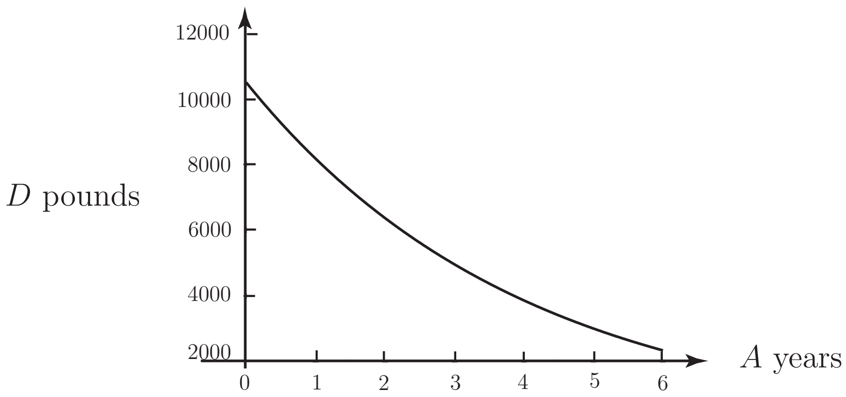


Figure 8: Plot of car depreciation over 6 years

Task

Produce the linearised model of $D = 10500e^{-0.25A}$.

Your solution

Answer

$\ln D = \ln 10500 + \ln(e^{-0.25A})$

so $\ln D = \ln 10500 - 0.25A$

Engineering Example 2

Exponential decay of sound intensity

Introduction

The rate at which a quantity decays is important in many branches of engineering and science. A particular example of this is exponential decay. Ideally the sound level in a room where there are substantial contributions from reflections at the walls, floor and ceiling will decay exponentially once the source of sound is stopped. The decay in the sound intensity is due to absorbtion of sound at the room surfaces and air absorption although the latter is significant only when the room is very large. The contributions from reflection are known as **reverberation**. A measurement of reverberation in a room of known volume and surface area can be used to indicate the amount of absorption.

Problem in words

As part of an emergency test of the acoustics of a concert hall during an orchestral rehearsal, consultants asked the principal trombone to play a single note at maximum volume. Once the sound had reached its maximum intensity the player stopped and the sound intensity was measured for the next 0.2 seconds at regular intervals of 0.02 seconds. The initial maximum intensity at time 0 was 1. The readings were as follows:

time	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
intensity	1	0.63	0.35	0.22	0.13	0.08	0.05	0.03	0.02	0.01	0.005

Draw a graph of intensity against time and, assuming that the relationship is exponential, find a function which expresses the relationship between intensity and time.

Mathematical statement of problem

If the relationship is exponential then it will be a function of the form

$$I = I_0 10^{kt}$$

and a log-linear graph of the values should lie on a straight line. Therefore we can plot the values and find the gradient and the intercept of the resulting straight-line graph in order to find the values for I_0 and k.

k is the gradient of the log-linear graph i.e.

$$k = \frac{\text{change in } \log_{10} \text{ (intensity)}}{\text{change in time}}$$

and I_0 is found from where the graph crosses the vertical axis $\log_{10}(I_0) = c$

Mathematical analysis

Figure 9(a) shows the graph of intensity against time.

We calculate the $\log_{10}$ (intensity) to create the table below:

time	0	0.02	0.04	0.06	0.08	0.10	0.12	0.14	0.16	0.18	0.20
$\log_{10}$(intensity)	0	-0.22	-0.46	-0.66	-0.89	-1.1	-1.3	-1.5	-1.7	-2.0	-2.2

Figure 9(b) shows the graph of $\log$ (intensity) against time.

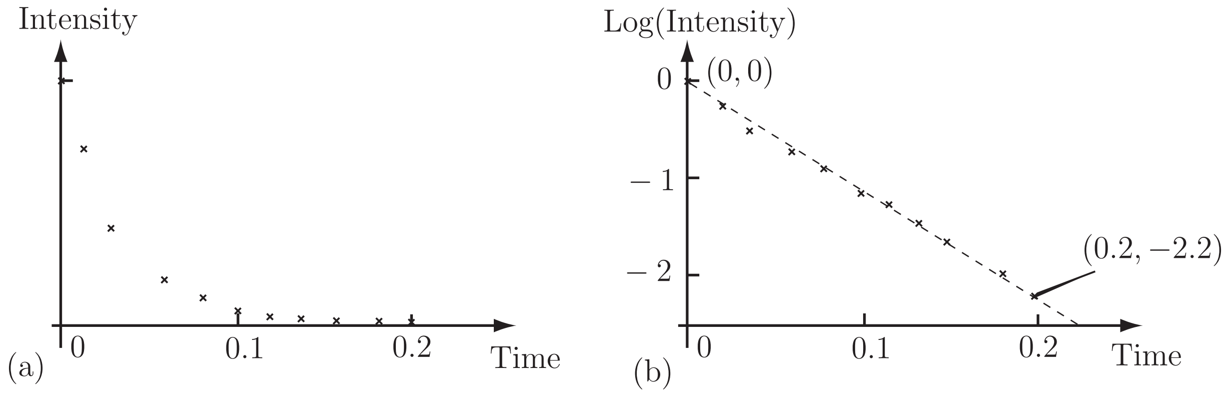


Figure 9: (a) Graph of sound intensity against time (b) Graph of $\log_{10}$ (intensity) against time and a line fitted by eye to the data. The line goes through the points $(0, 0)$ and $(0.2, -2.2)$.

We can see that the second graph is approximately a straight line and therefore we can assume that the relationship between the intensity and time is exponential and can be expressed as

$$I = I_0 10^{kt}.$$

The $\log_{10}$ of this gives

$$\log_{10}(I) = \log_{10}(I_0) + kt.$$

From the graph (b) we can measure the gradient, k using

$$k = \frac{\text{change in } \log_{10}\text{(intensity)}}{\text{change in time}}$$

giving $k = \dfrac{-2.2 - 0}{0.2 - 0} = -11$

The point at which it crosses the vertical axis gives

$$\log_{10}(I_0) = 0 \Rightarrow I_0 = 10^0 = 1$$

Therefore the expression $I = I_0 10^{kt}$ becomes

$$I = 10^{-11t}$$

Interpretation

The data recorded for the sound intensity fit exponential decaying with time. We have used a log-linear plot to obtain the approximate function:

$$I = 10^{-11t}$$

4. Growth and decay to a limit

Consider a function intended to represent the speed of a parachutist after the opening of the parachute where v m s^{-1} is the instantanous speed at time t s. An appropriate function is

$$v = 12 - 8e^{-1.25t} \qquad (t \geq 0),$$

We will look at some of the properties and modelling implications of this function. Consider first the value of v when $t = 0$:

$$v = 12 - 8e^0 = 12 - 8 = 4$$

This means the function predicts that the parachutist is moving at 4 m s^{-1} when the parachute opens. Consider next the value of v when t is arbitrarily large. For such a value of $t, 8e^{-1.25t}$ would be arbitrarily small, so v would be very close to the value 12. The modelling interpretation of this is that eventually the speed becomes very close to a constant value, 12 m s^{-1} which will be maintained until the parachutist lands.

The steady speed which is approached by the parachutist (or anything else falling against air resistance) is called the **terminal velocity**. The parachute, of course, is designed to ensure that the terminal velocity is sufficiently low (12 m s^{-1} in the specific case we have looked at here) to give a reasonably gentle landing and avoid injury.

Now consider what happens as t increases from near zero. When t is near zero, the speed will be near 4 m s^{-1}. The amount being subtracted from 12, through the term 8e$^{-1.25t}$, is close to 8 because $e^0 = 1$. As t increases the value of 8e$^{-1.25t}$ decreases fairly rapidly at first and then more gradually until v is very nearly 12. This is sketched in Figure 10. In fact v is never equal to 12 but gets imperceptibly close as anyone would like as t increases. The value shown as a horizontal broken line in Figure 10 is called an **asymptotic limit** for v.

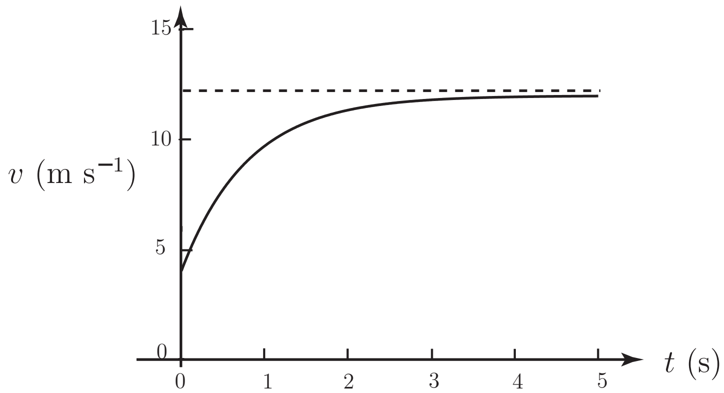


Figure 10: Graph of a parachutist's speed against time

The model concerned the approach of a parachutist's velocity to terminal velocity but the kind of behaviour portrayed by the resulting function is useful generally in modelling any **growth to a limit**. A general form of this type of growth-to-a-limit function is

$$y = a - be^{-kx} \qquad (C \leq x \leq D)$$

where a, b and k are positive constants (parameters) and C and D represent values of the independent variable between which the function is valid. We will now check on the properties of this general function. When $x = 0, y = a - be^0 = a - b$. As x increases the exponential factor e^{-kx} gets smaller, so y will increase from the value $a - b$ but at an ever-decreasing rate. As be^{-kx} becomes very small,

y, approaches the value a. This value represents the limit, towards which y grows. If a function of this general form was being used to create a model of population growth to a limit, then a would represent the limiting population, and $a - b$ would represent the starting population.

There are three parameters, a, b, and k in the general form. Knowledge of the initial and limiting population only gives two pieces of information. A value for the population at some non-zero time is needed also to evaluate the third parameter k.

As an example we will obtain a function to describe a food-limited bacterial culture that has 300 cells when first counted, has 600 cells after 30 minutes but seems to have approached a limit of 4000 cells after 18 hours.

We start by assuming the general form of growth-to-a-limit function for the bacteria population, with time measured in hours

$$P = a - be^{-kt} \qquad (0 \leq t \leq 18).$$

When $t = 0$ (the start of counting), $P = 300$. Since the general form gives $P = a - b$ when $t = 0$, this means that

$$a - b = 300.$$

The limit of P as t gets large, according to the general form $P = a - b^{-kt}$, is a, so $a = 4000$. From this and the value of $a - b$, we deduce that $b = 3700$. Finally, we use the information that $P = 600$ when t (measuring time in hours) $= 0.5$. Substitution in the general form gives

$$600 = 4000 - 3700e^{-0.5k}$$

$$3400 = 3700e^{-0.5k}$$

$$\frac{3400}{3700} = e^{-0.5k}$$

Taking natural logs of both sides:

$$\ln\left(\frac{3400}{3700}\right) = -0.5k \quad \text{so} \quad k = -2\ln(\frac{34}{37}) = 0.1691$$

Note, as a check, that k turns out to be positive as required for a growth-to-a-limit behaviour. Finally the required function may be written

$$P = 4000 - 3700e^{-0.1691t} \qquad (0 \leq t \leq 18).$$

As a check we should substitute $t = 18$ in this equation. The result is $P = 3824$ which is close to the required value of 4000.

Task

Find a function that could be used to model the growth of a population that has a value of 3000 when counts start, reaches a value of 6000 after 1 year but approaches a limit of 12000 after a period of 10 years.

(a) First find the modelling equation:

Your solution

Answer

Start with

$$P = a - be^{-kt} \qquad (0 \leq t \leq 10).$$

where P is the number of members of the population at time t years. The given data requires that a is 12000 and that $a - b = 3000$, so $b = 9000$.

The corresponding curve must pass through $(t = 1,\ P = 6000)$ so

$$6000 = 12000 - 9000e^{-k}$$

$$\mathrm{e}^{-k} = \frac{12000 - 6000}{9000} = \frac{2}{3} \qquad \text{so} \quad \mathrm{e}^{-kt} = (\mathrm{e}^{-k})^t = \left(\frac{2}{3}\right)^t$$ (using Rule 3b, Table 1, page 42)

So the population function is

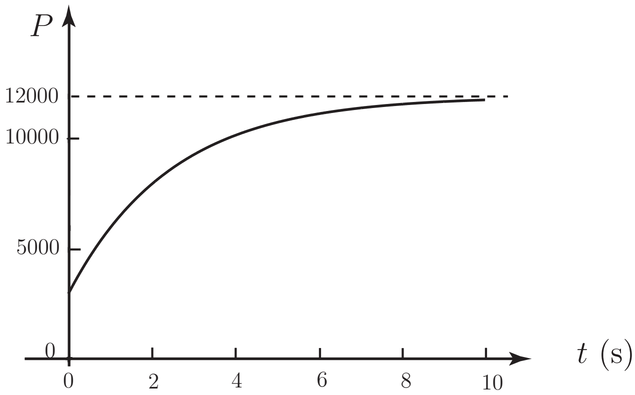
$$P = 12000 - 9000\left(\frac{2}{3}\right)^t \qquad (0 \leq t \leq 10).$$

Note that $P(10)$ according to this formula is approximately 11840, which is reasonably close to the required value of 12000.

(b) Now sketch this function:

Your solution

Answer



5. Inverse square law decay

Engineering Example 3

Inverse square law decay of electromagnetic power

Introduction

Engineers are concerned with using and intercepting many kinds of wave forms including electromagnetic, elastic and acoustic waves. In many situations the intensity of these signals decreases with the square of the distance. This is known as the **inverse square law**. The power received from a beacon antenna is expected to conform to the inverse square law with distance.

Problem in words

Check whether the data in the table below confirms that the measured power obeys this behaviour with distance.

Power received, W	0.393	0.092	0.042	0.021	0.013	0.008
Distance from antenna, m	1	2	3	4	5	6

Mathematical statement of problem

Represent power by P and distance by r. To show that the data fit the function $P = \dfrac{A}{r^2}$ where A is a constant, plot log(P) against log(r) (or plot the 'raw' data on log-log axes) and check

(a) how close the resulting graph is to that of a straight line

(b) how close the slope is to 2.

Mathematical analysis

The values corresponding to log(P) and log(r) are

log(P)	-0.428	-1.041	-1.399	-1.653	-1.851	-2.012
log(r)	0	0.301	0.499	0.602	0.694	0.778

These are plotted in Figure 11 and it is clear that they lie close to a straight line.

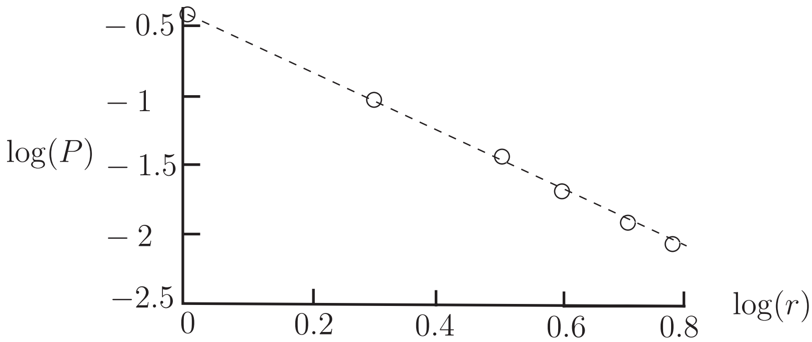


Figure 11

The slope of a line through the first and third points can be found from

$$\frac{-1.399 - (-0.428)}{0.499 - 0} = -2.035$$

The negative value means that the line slopes downwards for increasing r. It would have been possible to use any pair of points to obtain a suitable line but note that the last point is least 'in line' with the others. Taking logarithms of the equation $P = \dfrac{A}{r^n}$ gives $\log(P) = \log(A) - n\log(r)$

The inverse square law corresponds to $n = 2$. In this case the data yield $n = 2.035 \approx 2$. Where log(r) $= 0$, log(P) $=$ log(A). This means that the intercept of the line with the log(P) axis gives the value of log(A) $= -0.428$. So $A = 10 - 0.428 = 0.393$.

Interpretation

If the power decreases with distance according to the inverse square law, then the slope of the line should be -2. The calculated value of $n = 2.035$ is sufficiently close to confirm the inverse square law. The values of A and n calculated from the data imply that P varies with r according to

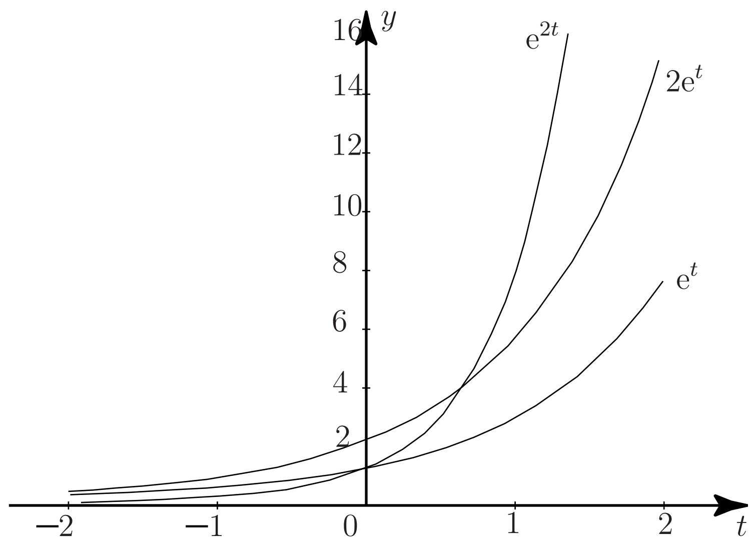
$$P = \frac{0.4}{r^2}$$

The slope of the line on a log-log plot is a little larger than -2. Moreover the points at 5 m and 6 m range fall below the line so there may be additional attenuation of the power with distance compared with predictions of the inverse square law.

Exercises

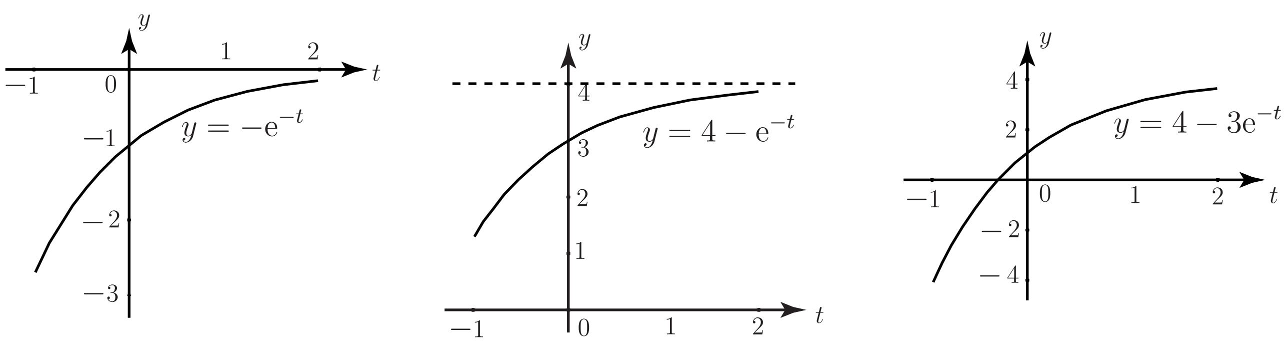
1. Sketch the graphs of (a) $y = \mathrm{e}^t$ (b) $y = \mathrm{e}^t + 3$ (c) $y = \mathrm{e}^{-t}$ (d) $y = \mathrm{e}^{-t} - 1$

2. The figure below shows the graphs of $y = \mathrm{e}^t$, $y = 2\mathrm{e}^t$ and $y = \mathrm{e}^{2t}$.



State in words how the graphs of $y = 2\mathrm{e}^t$ and $y = \mathrm{e}^{2t}$ relate to the graph of $y = \mathrm{e}^t$.

3. The figures below show graphs of $y = -\mathrm{e}^{-t}$, $y = 4 - \mathrm{e}^{-t}$ and $y = 4 - 3\mathrm{e}^{-t}$.



Use the above graphs to help you to sketch graphs of (a) $y = 5 - \mathrm{e}^{-t}$ (b) $y = 5 - 2\mathrm{e}^{-t}$

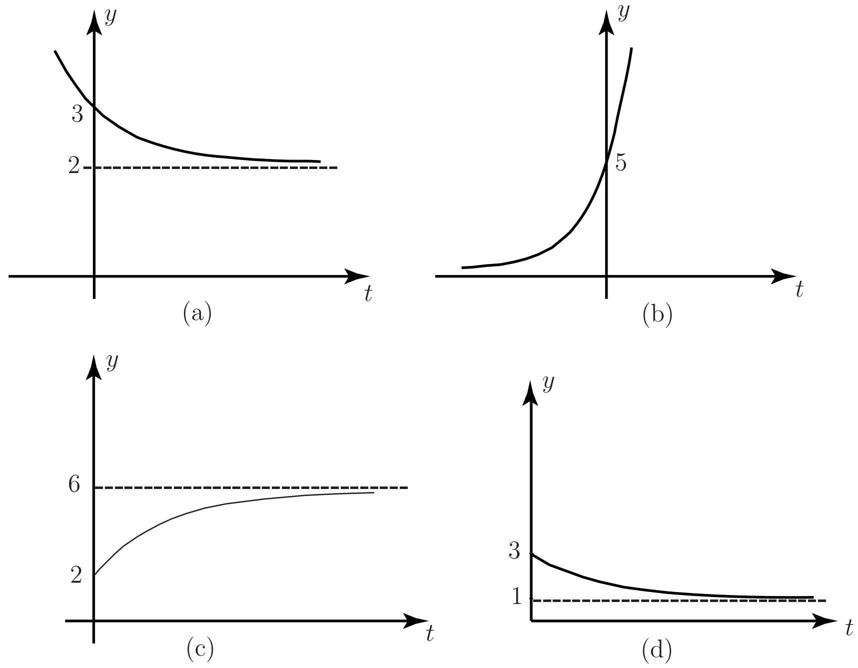
4. (a) The graph (a) in the figure below has an equation of the form

   $y = A + \mathrm{e}^{-kt}$, where A and k are constants. What is the value of A?

   (b) The graph (b) below has an equation of the form $y = A\mathrm{e}^{kt}$ where A and k are constants. What is the value of A?

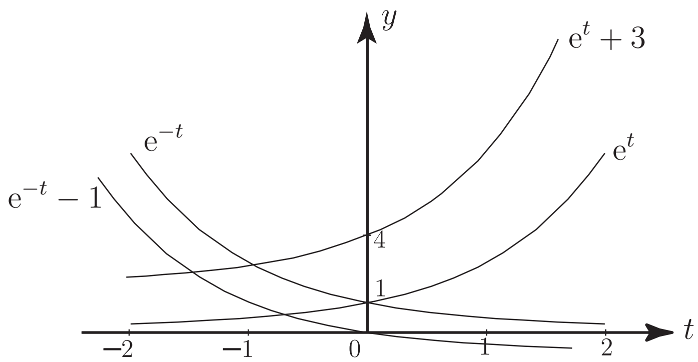
   (c) Write down a possible form of the equation of the exponential graph (c) giving numerical values to as many constants as possible.

   (d) Write down a possible form of the equation of the exponential graph (d) giving numerical values to as many constants as possible.



Answers

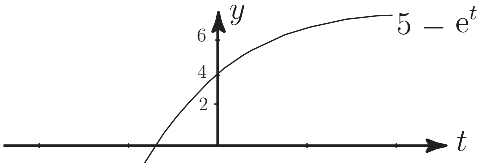
1.



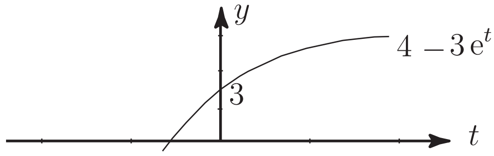
2. (a) $y = 2\text{e}^t$ is the same shape as $y = \text{e}^t$ but with all y values doubled.

   (b) $y = \text{e}^{2t}$ is much steeper than $y = \text{e}^t$ for $t > 0$ and much flatter for $t < 0$. Both pass through $(0, 1)$. Note that $y = \text{e}^{2t} = (\text{e}^t)^2$ so each value of $y = \text{e}^{2t}$ is the square of the corresponding value of $y = \text{e}^t$.

3. (a)



(b)



4. (a) 2 (b) 5 (c) $y = 6 - 4\text{e}^{-kt}$ (d) $y = 1 + 2\text{e}^{-kt}$

6. Logarithmic relationships

Experimental psychology is concerned with observing and measuring human response to various stimuli. In particular, sensations of light, colour, sound, taste, touch and muscular tension are produced when an external stimulus acts on the associated sense. A nineteenth century German, Ernst Weber, conducted experiments involving sensations of heat, light and sound and associated stimuli. Weber measured the response of subjects, in a laboratory setting, to input stimuli measured in terms of energy or some other physical attribute and discovered that:

(1) No sensation is felt until the stimulus reaches a certain value, known as the threshold value.

(2) After this threshold is reached an increase in stimulus produces an increase in sensation.

(3) This increase in sensation occurs at a diminishing rate as the stimulus is increased.

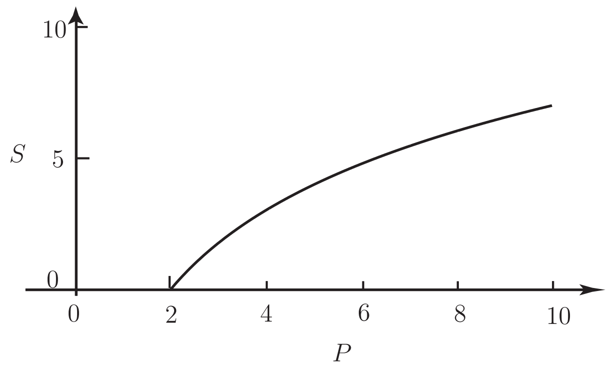
Task

(a) Do Weber's results suggest a linear or non-linear relationship between sensation and stimulus? Sketch a graph of sensation against stimulus according to Weber's results.

(b) Consider whether an exponential function or a growth-to-a-limit function might be an appropriate model.

Answer

(a) Non-linearity is required by observation (3).



(b) An exponential-type of growth is not appropriate for a model consistent with these experimental results, since we need a diminishing rate of growth in sensation as the stimulus increases. A growth-to-a-limit type of function is not appropriate since the data, at least over the range of Weber's experiments, do not suggest that there is a limit to the sensation with continuing increase in stimulus; only that the increase in sensation occurs more and more slowly.

A late nineteenth century German scientist, Gustav Fechner, studied Weber's results. Fechner suggested that an appropriate function modelling Weber's findings would be logarithmic. He suggested that the variation in sensation (S) with the stimulus input (P) is modelled by

$$S = A\log(P/T) \qquad (0 < T \le 1)$$

where T represents the threshold of stimulus input below which there is no sensation and A is a constant. Note that when $P = T, \log(P/T) = \log(1) = 0$, so this function is consistent with item (1) of Weber's results. Recall also that $\log$ means logarithm to base 10, so when $P = 10T, S = A\log(10) = A$. When P = 100T, $S = A\log(100) = 2A$. The logarithmic function predicts that a tenfold increase in the stimulus input from T to $10T$ will result in the same change in sensation as a further tenfold increase in stimulus input to $100T$. Each tenfold change is stimulus results in a doubling of sensation. So, although sensation is predicted to increase with stimulus, the stimulus has to increase at a faster and faster rate (i.e. exponentially) to achieve a given change in sensation. These points are consistent with items (2) and (3) of Weber's findings. Fechner's suggestion, that the logarithmic function is an appropriate one for a model of the relationship between sensation and stimulus, seems reasonable. Note that the logarithmic function suggested by Weber is not defined for zero stimulus but we are only interested in the model at and above the threshold stimulus, i.e. for values of the logarithm equal to and above zero. Note also that the logarithmic function is useful for looking at changes in sensation relative to stimulus values other than the threshold stimulus. According to Rule 2a in Table 2 on page 42, Fechner's sensation function may be written

$$S = A\log(P/T) = A[\log(P) - \log(T)] \qquad (P \ge T > 0).$$

Suppose that the sensation has the value S_1 at P_1 and S_2 at P_2, so that

$$S_1 = A[\log(P_1) - \log(T)] \qquad (P_1 \ge T > 0),$$

and

$$S_2 = A[\log(P_2) - \log(T)] \qquad (P_2 \ge T > 0).$$

If we subtract the first of these two equations from the second, we get

$$S_2 - S_1 = A[\log(P_2) - \log(P_1)] = A\log(P_2/P_1),$$

where Rule 2a of Table 2 has been used again for the last step. According to this form of equation, the change in sensation between two stimuli values depends on the ratio of the stimuli values.
We start with

$$S = A\log(P/T) \qquad (1 \ge T > 0).$$

Divide both sides by A:

$$\frac{S}{A} = \log\frac{P}{T} \qquad (1 \ge T > 0).$$

'Undo' the logarithm on both sides by raising 10 to the power of each side:

$$10^{S/A} = 10^{\log(P/T)} = \frac{P}{T} \qquad (1 \ge T > 0),$$ using Rule 4b of Table 2.

So $P = T \times 10^{S/A} \qquad (1 \ge T > 0)$ which is an exponential relationship between stimulus and sensation.

A **logarithmic** relationship between sensation and stimulus therefore implies an **exponential** relationship between stimulus and sensation. The relationship may be written in two different forms with the variables playing opposite roles in the two functions.

The logarithmic relationship between sensation and stimulus is known as the *Weber-Fechner Law of Sensation.* The idea that a mathematical function could describe our sensations was startling when first propounded. Indeed it may seem quite amazing to you now. Moreover it doesn't always work. Nevertheless the idea has been quite fruitful. Out of it has come much quantitative experimental psychology of interest to sound engineers. For example, it relates to the sensation of the loudness of sound. Sound level is expressed on a logarithmic scale. At a frequency of 1 kHz an increase of 10 dB corresponds to a doubling of loudness.

Task

Given a relationship between y and x of the form $y = 3\log(\frac{x}{4}) \qquad (x \geq 4)$, find the relationship between x and y.

Your solution

Answer

One way of answering is to compare with the example preceding this task. We have y in place of S, x in place of P, 3 in place of A, 4 in place of T. So it is possible to write down immediately

$$x = 4 \times 10^{y/3} \qquad (y \geq 0)$$

Alternatively we can manipulate the given expression algebraically.

Starting with $y = 3\log(x/4)$, divide both sides by 3 to give $y/3 = \log(x/4)$.

Raise 10 to the power of each side to eliminate the log, so that $10^{y/3} = x/4$.

Multiply both sides by 4 and rearrange, to obtain $x = 4 \times 10^{y/3}$, as before.

The associated range is the result of the fact that $x \geq 4$, so $10^{y/3} \geq 1$, so $y/3 > 0$ which means $y > 0$.

Log-linear Graphs

6.6

Introduction

In this Section we employ our knowledge of logarithms to simplify plotting the relation between one variable and another. In particular we consider those situations in which one of the variables requires scaling because the range of its data values is very large in comparison to the range of the other variable.

We will only employ logarithms to base 10. To aid the plotting process we explain how log-linear graph paper is used. Unlike ordinary graph paper, one of the axes is scaled using logarithmic values instead of the values themselves. By this process, values which range from (say) 1 to 1,000,000 are scaled down to range over the values 0 to 6. We do not discuss log-log graphs, in which both data sets require scaling, as the reader will easily be able to adapt the technique described here to those situations.

Prerequisites

Before starting this Section you should ...

- be familiar with the laws of logarithms
- have knowledge of logarithms to base 10
- be able to solve equations involving logarithms

Learning Outcomes

On completion you should be able to ...

- decide when to use log-linear graph paper
- use log-linear graph paper to analyse functions of the form $y = ka^{px}$

1. Logarithms and scaling

In this Section we shall work entirely with logarithms to base 10.

We are already familiar with a particular property of logarithms: $\log A^k = k \log A$.

Now, choosing $A = 10$ we see that: $\log 10^k = k \log 10 = k$.

The effect of taking a logarithm is to replace a power: 10^k (which could be very large) by the value of the exponent k. Thus a range of numbers extending from 1 to 1,000,000 say, can be transformed, by taking logarithms to base 10, into a range of numbers from 0 to 6. This approach is especially useful in the exercise of plotting one variable against another in which one of the variables has a wide range of values.

Example 10

Plot the following values (x, y)

x	1.0	1.1	1.2	1.3	1.4	1.5	1.6
y	1.0	2.14	4.3	8.16	14.8	25.6	42.9

Estimate the value of y when $x = 1.35$.

Solution

If we attempt to plot these values on ordinary graph paper in which both vertical and horizontal scales are linear we find the large range in the y-values presents a problem. The values near the lower end are bunched together and interpolating to find the value of y when $x = 1.35$ is difficult.

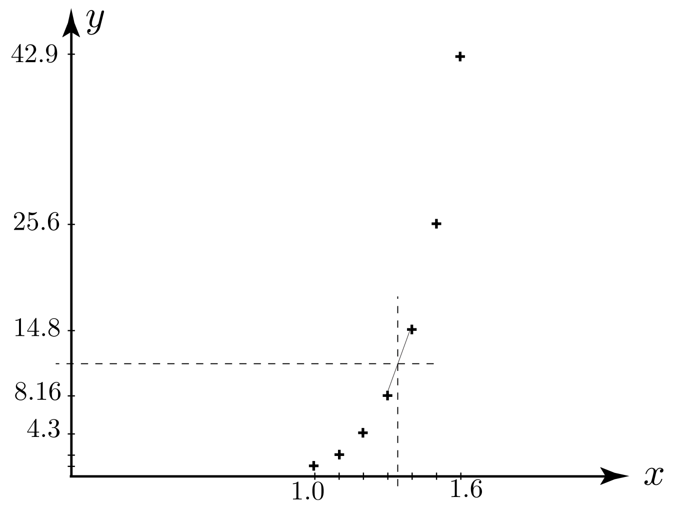


Figure 12

Example 11

To alleviate the scaling problem in Example 10 employ logarithms to scale down the y-values, giving:

x	1	1.1	1.2	1.3	1.4	1.5	1.6
$\log y$	0	0.33	0.63	0.97	1.17	1.41	1.63

Plot these values and estimate the value of y when $x = 1.35$.

Solution

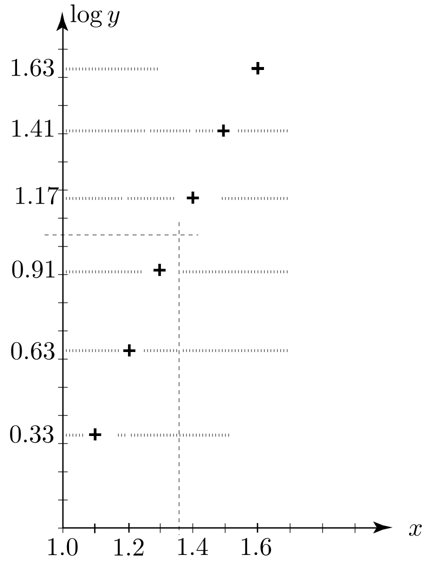


Figure 13

This approach has spaced-out the vertical values allowing a much easier assessment for the value of y at $x = 1.35$. From the graph we see that at $x = 1.35$ the '$\log y$' value is approximately 1.05. Taking $\log y = 1.05$ and inverting we get

$$y = 10^{1.05} = 11.22$$

2. Log-linear graph paper

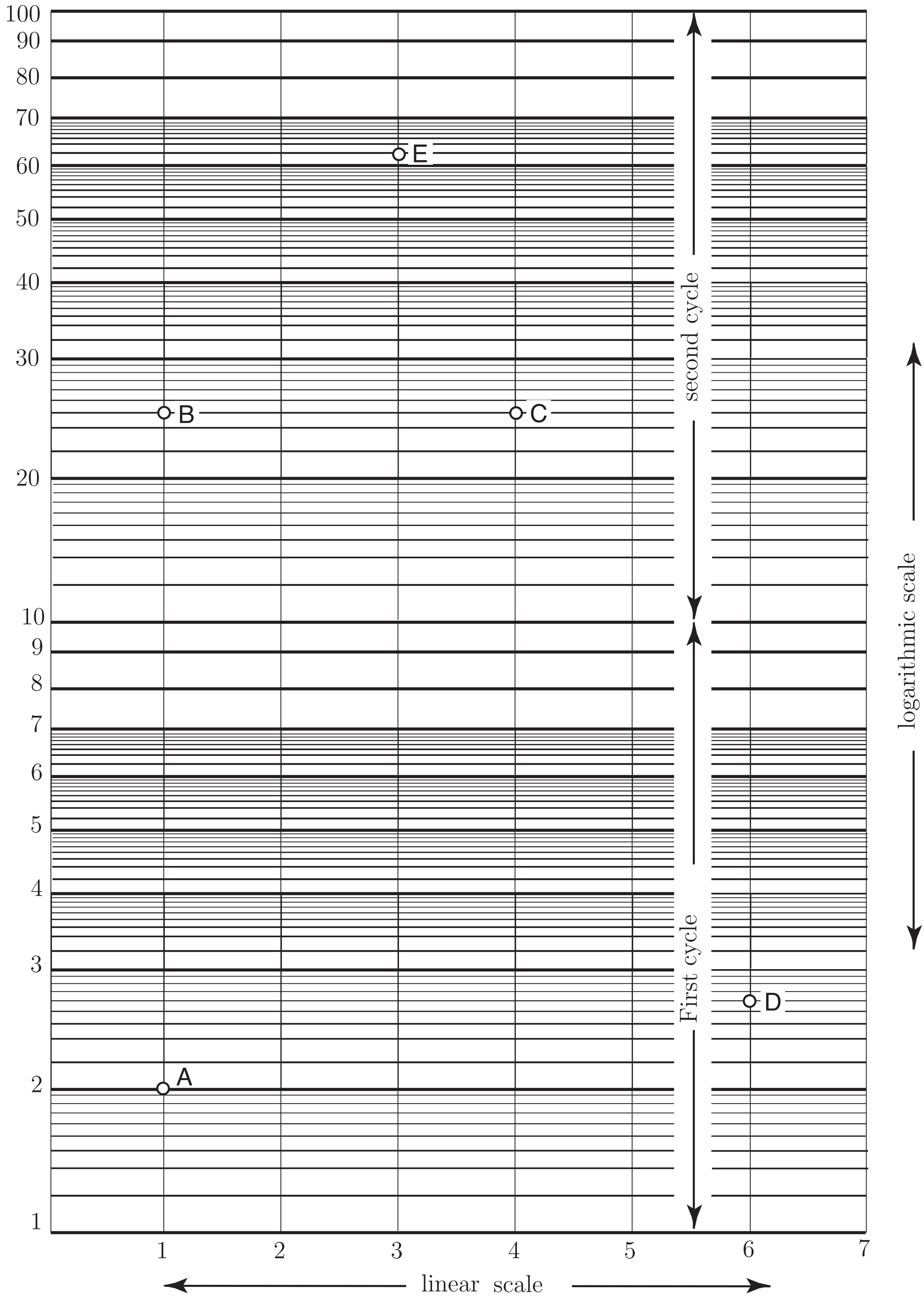
Ordinary graph paper has **linear** scales in both the horizontal (x) and vertical (y) directions. As we have seen, this can pose problems if the range of one of the variables, y say, is very large. One way round this is to take the **logarithm** of the y-values and re-plot on ordinary graph paper. Another common approach is to use **log-linear graph paper** in which the vertical scale is a **non-linear logarithmic scale**. Use of this special graph paper means that the original data can be plotted directly without the need to convert to logarithms which saves time and effort.

In log-linear graph paper the vertical axis is divided into a number of **cycles**. Each cycle corresponds to a jump in the data values by a factor of 10. For example, if the range of y-values extends from (say) 1 to 100 (or equivalently 10^0 to 10^2) then 2-cycle log-linear paper would be required. If the y-values extends from (say) 100 to 100,000 (or equivalently from 10^2 to 10^5) then 3-cycle log-linear paper would be used. Some other examples are given in Table 3:

Table 3

y − values	$\log y$ values	no. of cycles
$1 \to 10$	$0 \to 1$	1
$1 \to 100$	$0 \to 2$	2
$10 \to 10,000$	$1 \to 4$	3
$\frac{1}{10} \to 100$	$-1 \to 2$	3

An example of 2-cycle log-linear graph paper is shown in Figure 14. We see that the horizontal scale is linear. The vertical scale is divided by lines denoted by 1,2,3,. . . ,10,20,30,. . . ,100. In the first cycle each of the horizontal blocks (separated by a slightly thicker line) is also divided according to a log-linear scale; so, for example, in the range $1 \to 2$ we have 9 horizontal lines representing the values 1.1, 1.2, . . . , 1.9. These subdivisions have been repeated (appropriately scaled) in blocks 2-3, 3-4, 4-5, 5-6, 6-7. The subdivisions have been omitted from blocks 7-8, 8-9, 9-10 for reasons of clarity. On this graph paper, we have noted the positions of $A:(1,2),\ B:(1,23),\ C:(4,23),\ D:(6,2.5),\ E:(3,61)$.


100
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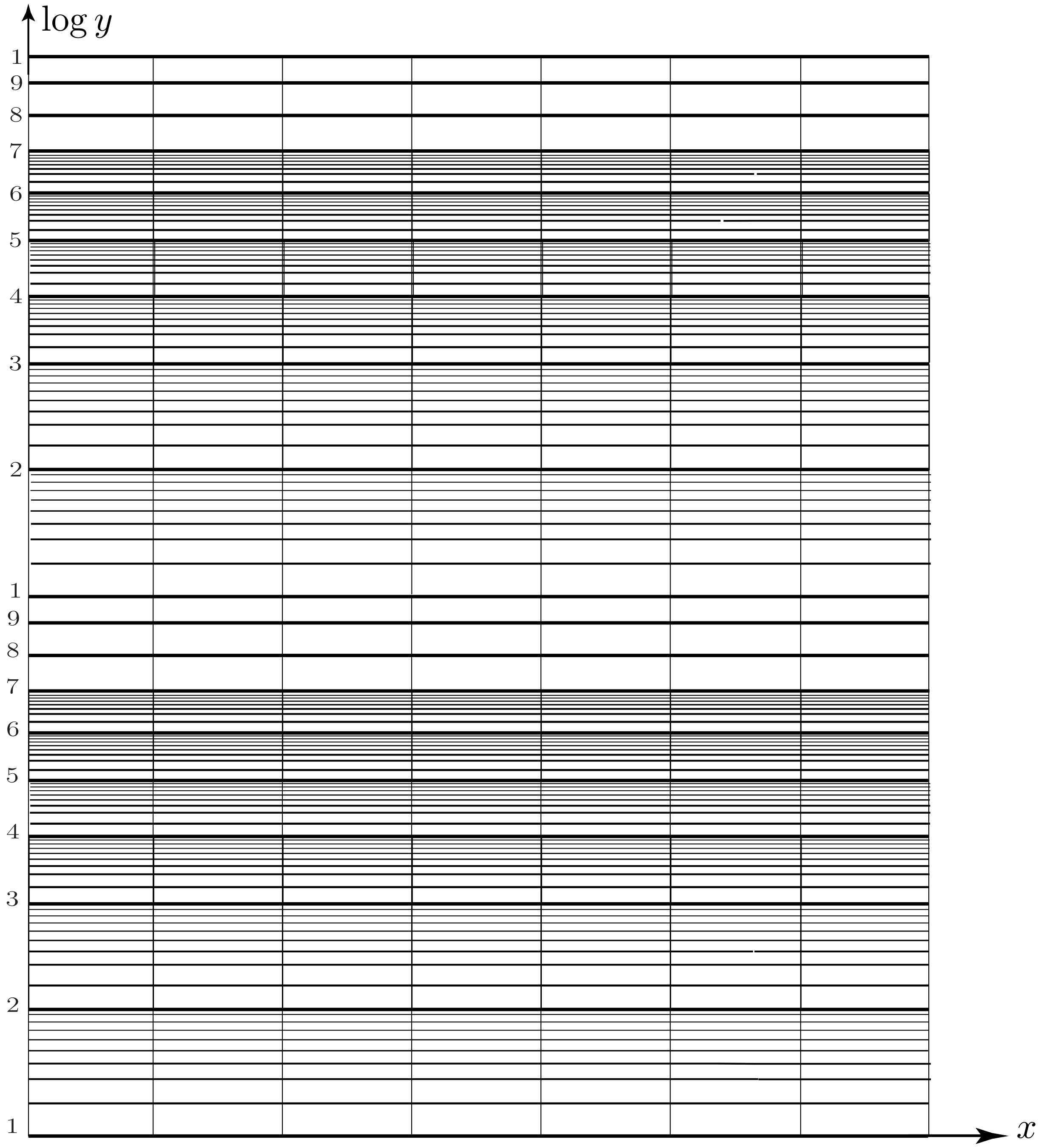
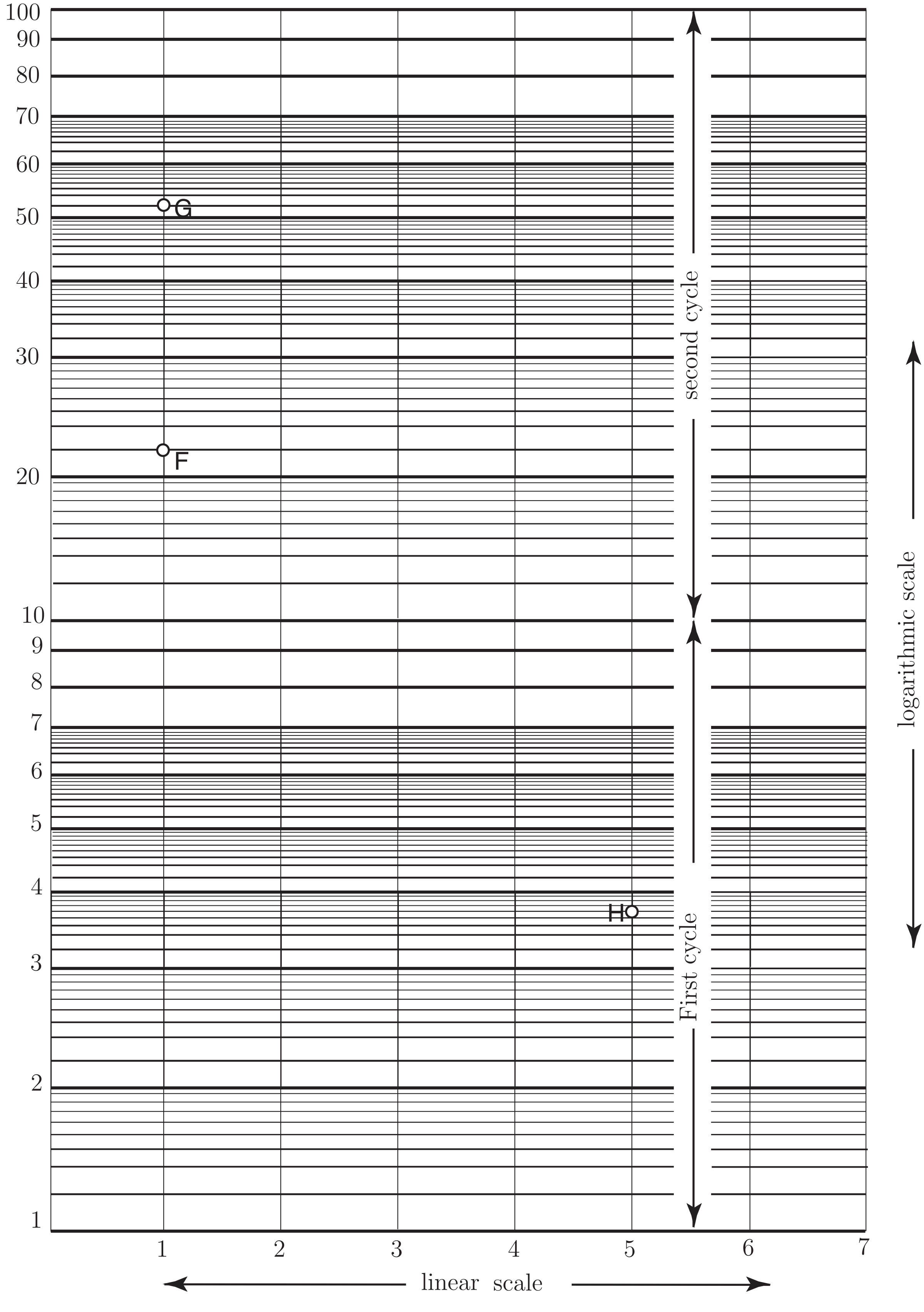
3
2
1
E
B
C
D
A
second cycle
First cycle
logarithmic scale
1
2
3
4
5
6
7
linear scale

Figure 14

Task

On the 2-cycle log-linear graph paper (below) locate the positions of the points $F : (2, 21)$, $G : (2, 51)$, $H : (5, 3.5)$. [The correct positions are shown on the graph on next page.]




100
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7
6
5
4
3
2
1
G
F
H
second cycle
First cycle
logarithmic scale
1
2
3
4
5
6
7
linear scale

Example 12

It is thought that the relationship between two variables x, y is exponential

$$y = ka^x$$

An experiment is performed and the following pairs of data values (x, y) were obtained

x	1	2	3	4	5
y	5.9	12	26	49	96

Verify that the relation $y = ka^x$ is valid by plotting values on log-linear paper to obtain a set of points lying on a straight line. Estimate the values of k, a.

Solution

First we rearrange the relation $y = ka^x$ by taking logarithms (to base 10).

$$\therefore \qquad \log y = \log(ka^x) = \log k + x \log a$$

So, if we define a new variable $Y \equiv \log y$ then the relationship between Y and x will be linear – its graph (on log-linear paper) should be a straight line. The vertical intercept of this line is $\log k$ and the gradient of the line is $\log a$. Each of these can be obtained from the graph and the values of a, k inferred.

When using log-linear graphs, the reader should keep in mind that, on the vertical axis, the values are not as written but the logarithms of those values.

We have plotted the points and drawn a straight line (as best we can) through them - see Figure 15. (We will see in a later Workbook (HELM 31) how we might improve on this subjective approach to fitting straight lines to data points). The line intersects the vertical axis at a value $\log(3.13)$ and the gradient of the line is

$$\frac{\log 96 - \log 3.13}{5 - 0} = \frac{\log(96/3.13)}{5} = \frac{\log 30.67}{5} = 0.297$$

But the intercept is $\log k$ so

$$\log k = \log 3.13 \qquad \text{implying} \qquad k = 3.13$$

and the gradient is $\log a$ so

$$\log a = 0.297 \qquad \text{implying} \qquad a = 10^{0.297} = 1.98$$

We conclude that the relation between the x, y variables is well modelled by the relation $y = 3.13(1.98)^x$. If the points did not lie more-or-less on a straight line then we would conclude that the relationship was *not* of the form $y = ka^x$.

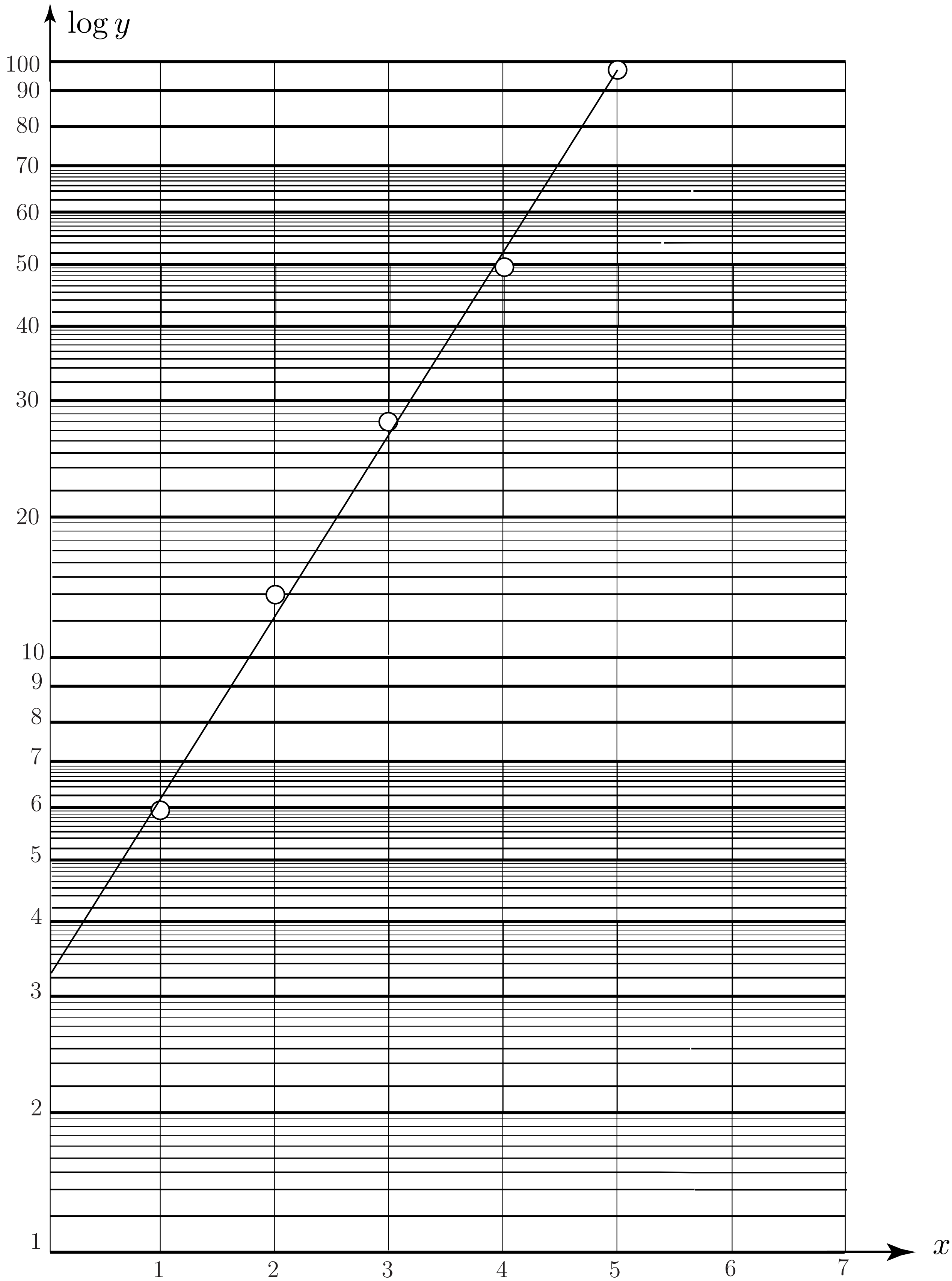


Figure 15

Task

Using a log-linear graph estimate the values of k, a if it is assumed that $y = ka^{-2x}$ and the data values connecting x, y are:

x	-0.3	-0.2	-0.1	0.0	0.1	0.2	0.3
y	190	155	123	100	80	63	52

First take logs of the relation $y = ka^{-2x}$ and introduce an appropriate new variable:

Your solution

$y = ka^{-2x}$ implies $\log y = \log(ka^{-2x}) =$

introduce $Y =$

$\log y = \log k - 2x \log a$. Let $Y = \log y$ then $Y = \log k + x(-2\log a)$. We therefore expect a linear relation between Y and x (i.e. on log-linear paper).

Now determine how many cycles are required in your log-linear paper:

Your solution

The range of values of y is 140; from 5.2×10 to 1.9×10^2. So 2-cycle log-linear paper is needed.

Now plot the data values directly onto log-linear paper (supplied on the next page) and decide whether the relation $y = ka^{-2x}$ is acceptable:

Your solution

It is acceptable. On plotting the points a straight line fits the data well which is what we expect from $Y = \log k + x(-2\log a)$.

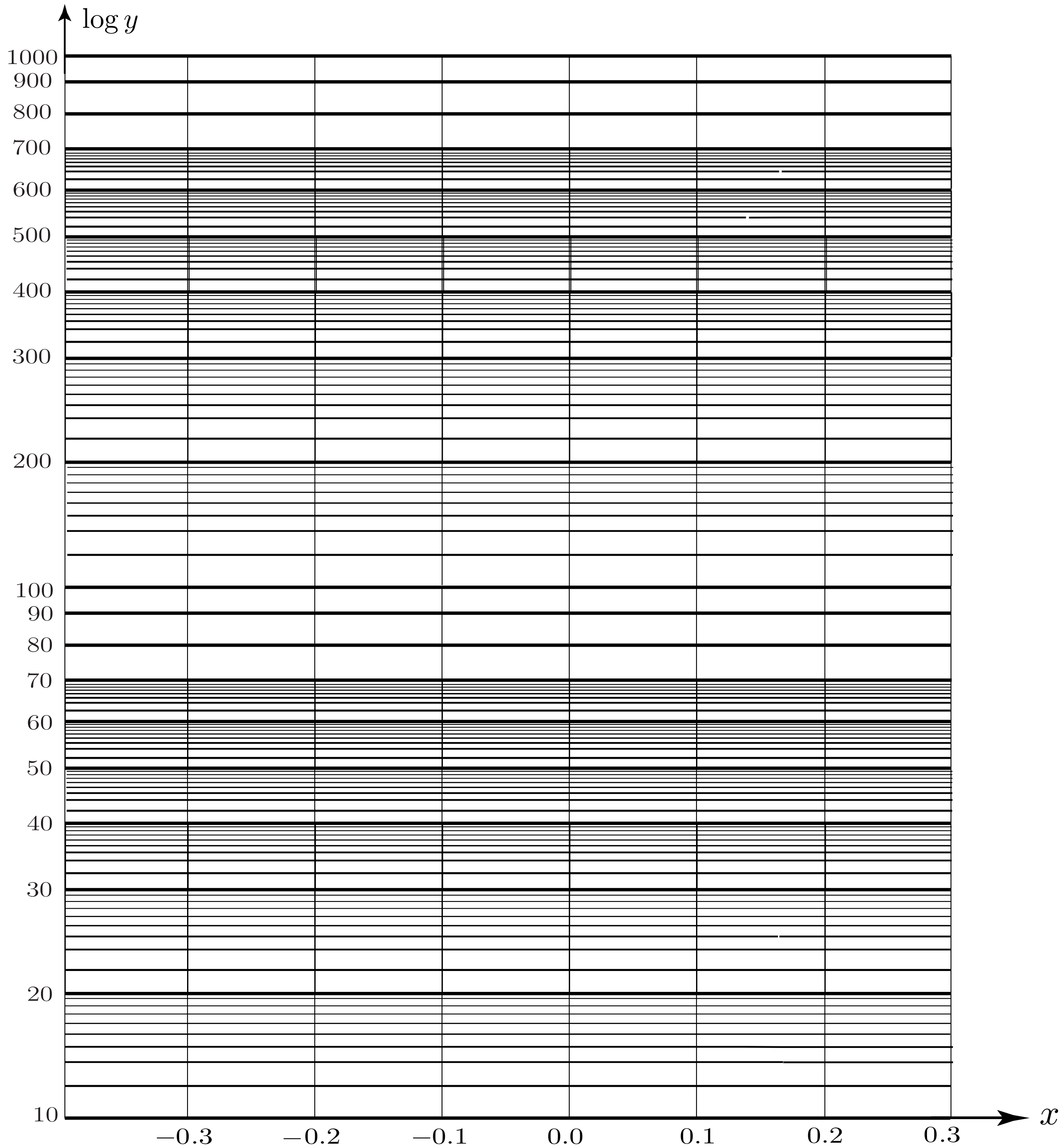
Now, using knowledge of the intercept and the gradient, find the values of k, a:

Your solution

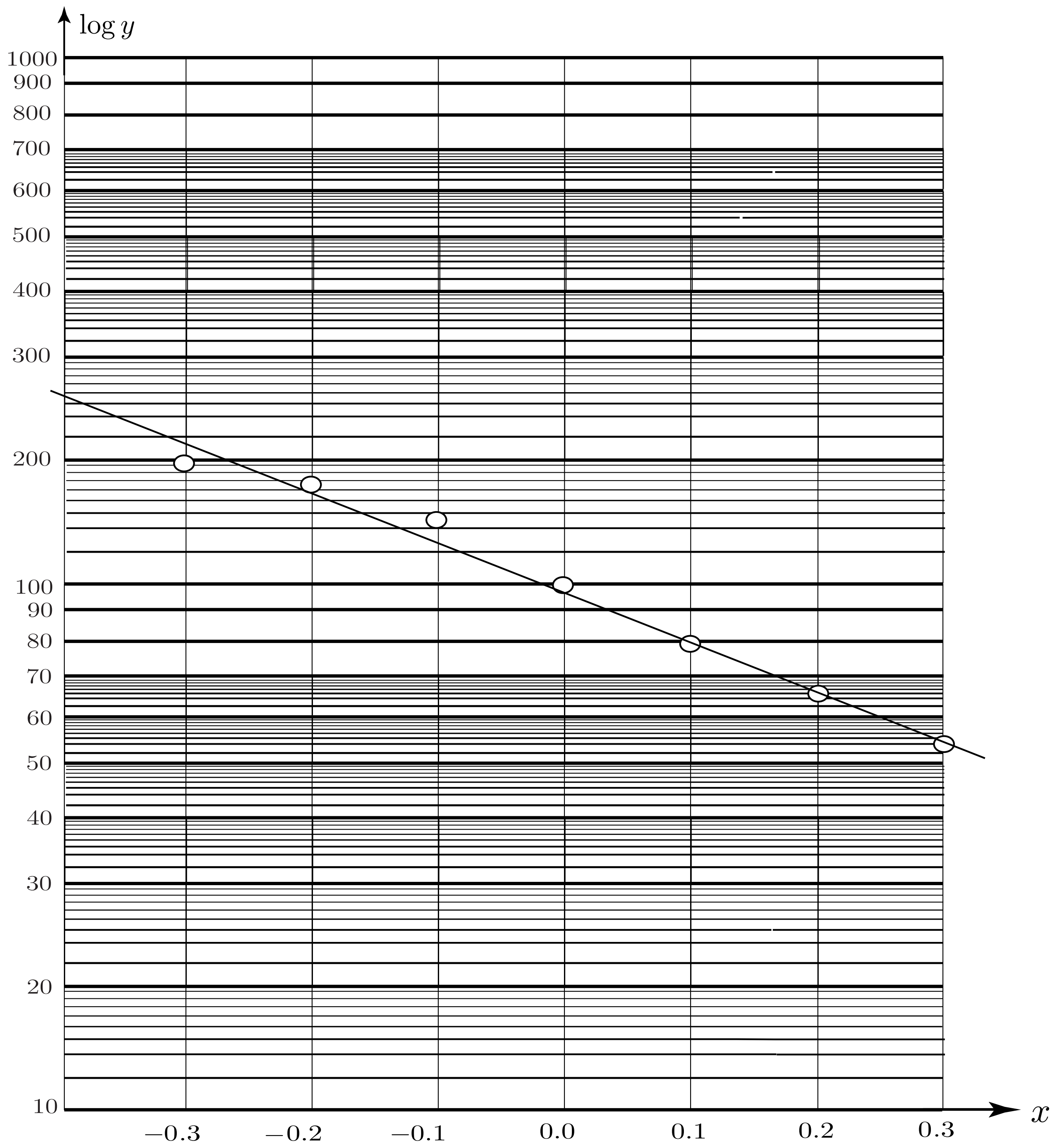
See the graph two pages further on. $k \approx 94$ (intercept on $x = 0$ line). The gradient is

$$\frac{\log 235 - \log 52}{-0.4 - 0.3} = -\frac{\log(235/52)}{0.7} = -\frac{0.655}{0.7} = -0.935$$

But the gradient is $-2\log a$. Thus $-2\log a = -0.935$ which implies $a = 10^{0.468} = 2.93$



Your solution to Task on page 67



Answer to Task on page 67

Use the log-linear graph sheets supplied on the following pages for these Exercises.

Exercises

1. Estimate the values of k and a if $y = ka^x$ represents the following set of data values:

x	0.5	1	2	3	4
y	5.93	8.8	19.36	42.59	93.70

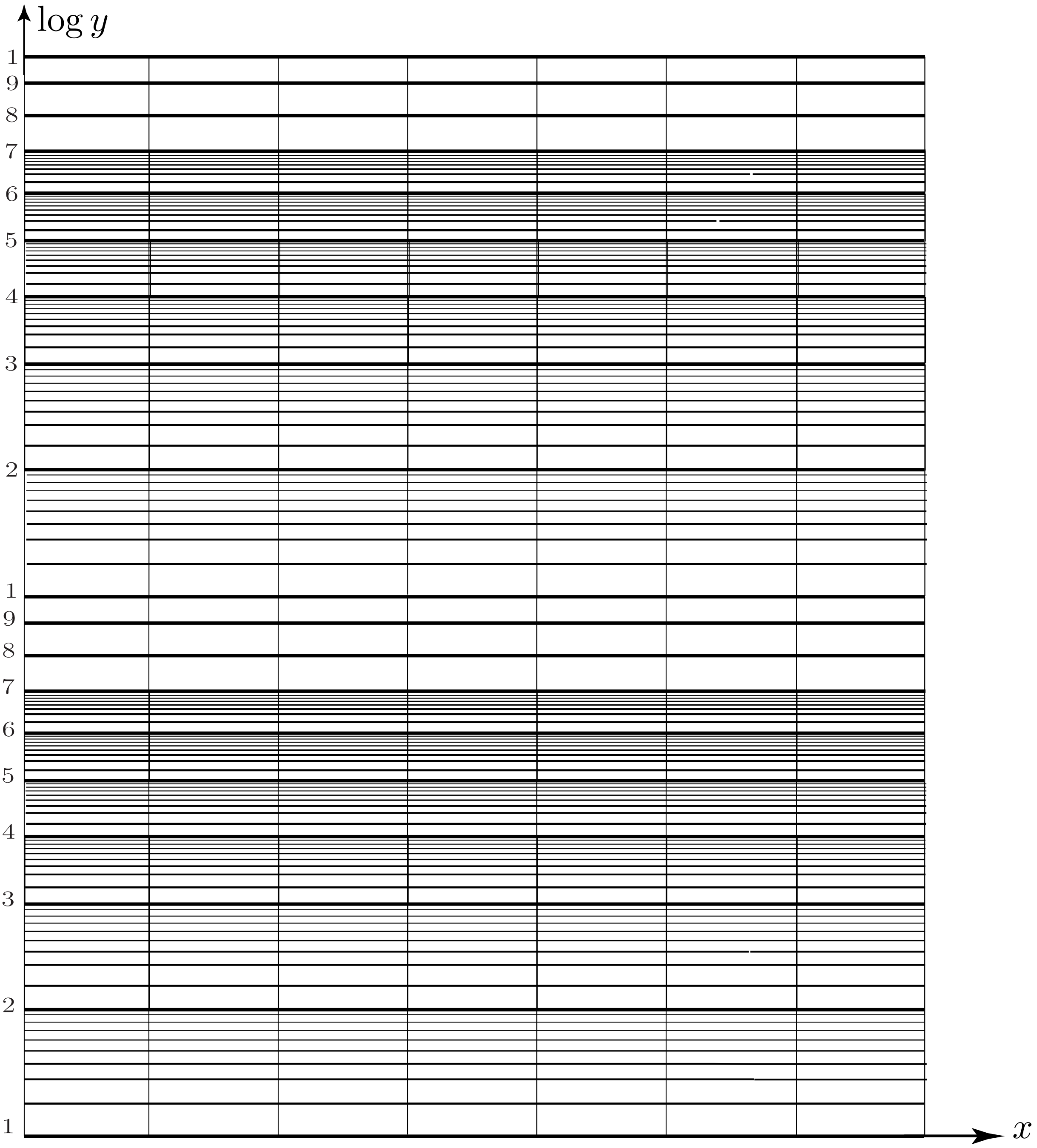
2. Estimate the values of k and a if the relation $y = k(a)^{-x}$ is a good representation for the data values:

x	2	2.5	3	3.5	4
y	7.9	3.6	1.6	0.7	0.3

Answers

1. $k \approx 4 \quad a \approx 2.2$

2. $k \approx 200 \quad a \approx 5$


log y
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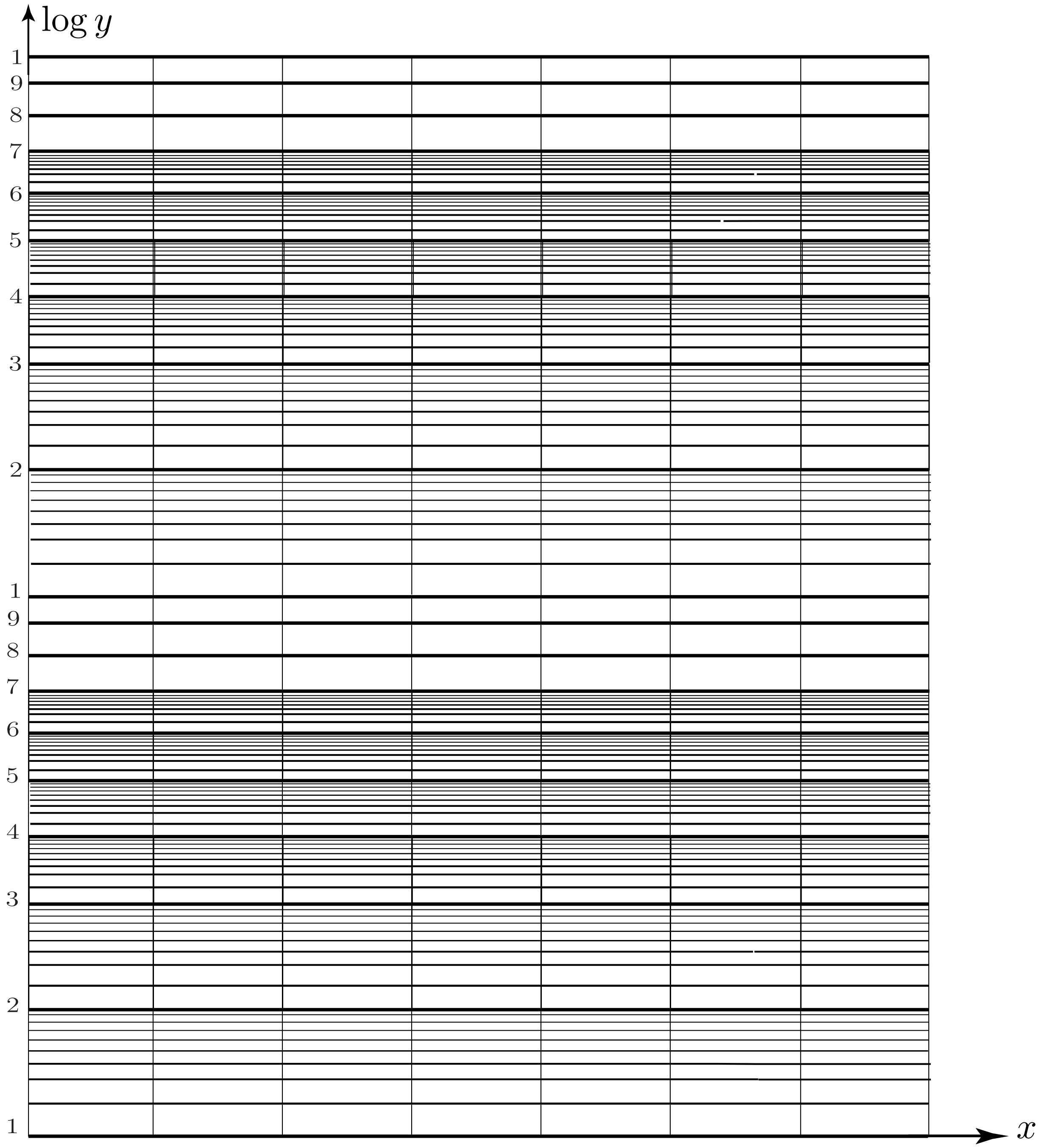
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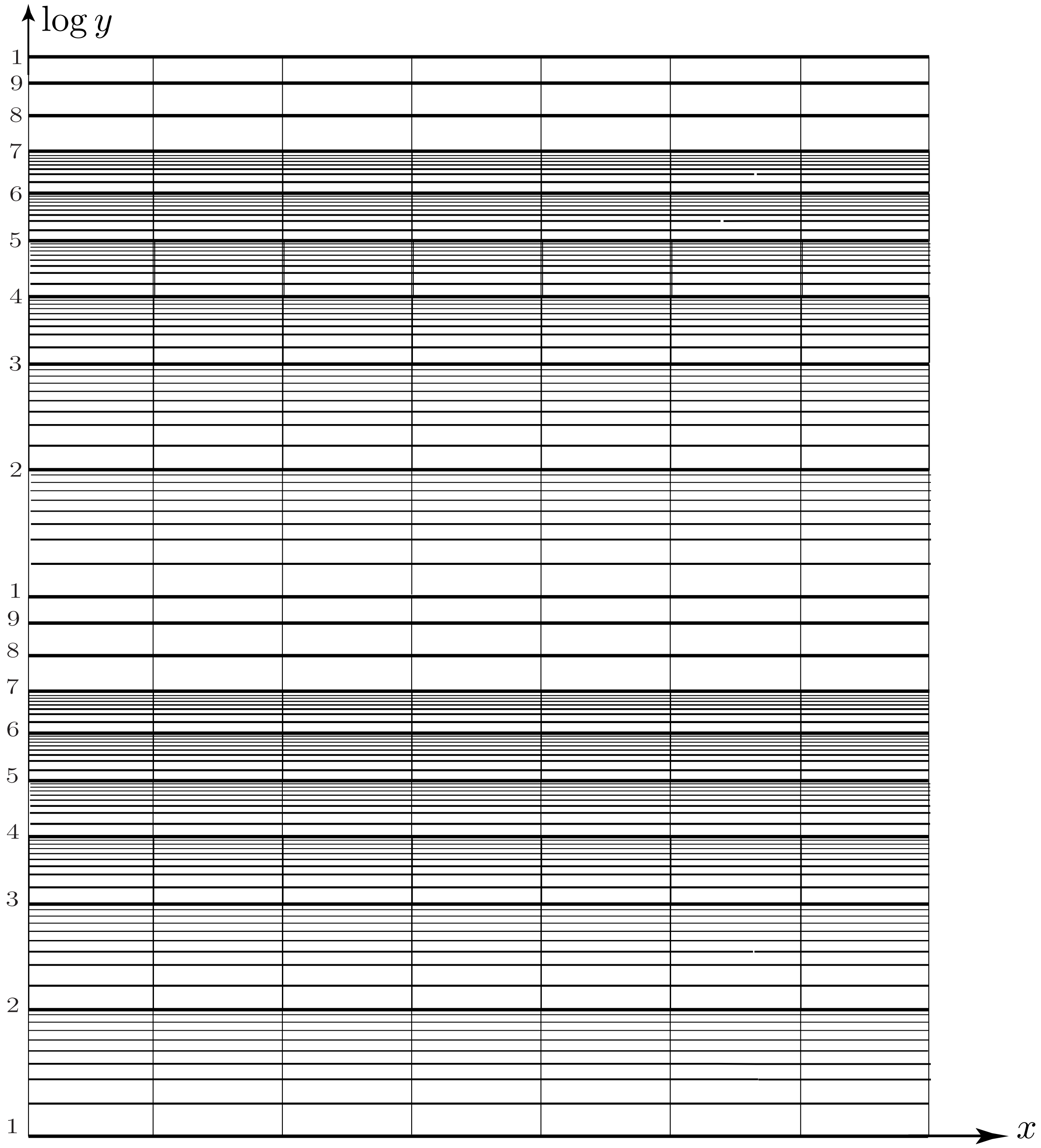
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